Euler's Theorem

Introduction

This is how I was shown Euler's Theorem when studying at Rugby.

Suppose $y = \cos\theta + i\sin\theta$

then $dy / d\theta = -\sin\theta + i\cos\theta$

$dy / d\theta = i\,y$

$dy / y = i\,d\theta$

ie $\int dy / y = \int i\,d\theta$

$\ln y = i\theta + k$

let $\theta = 0$ in the original expression

then $y = \cos 0 = 1$ so $k = 0$

Hence $\ln y = i\theta$

$y = e^{i\theta}$

So $e^{i\theta} = \cos\theta + i\sin\theta$

which is a pretty neat way of deriving such a key relationship.

Set $\theta = \pi$ and $e^{i\pi} = 1$ or $e^{i\pi} - 1 = 0$

Expansion of $e^{i\theta}$

We can now substitute $i\theta$ into the expansion of e^x.

$$e^{i\theta} = 1 + i\theta + i^2\theta^2/2! + i^3\theta^3/3! \ldots$$
$$= 1 + i\theta - \theta^2/2! - i\theta^3/3! \ldots$$
$$= 1 + - \theta^2/2! - \theta^4/4! \ldots$$
$$+ i\,(\theta - \theta^3/3! + \theta^5/5! \ldots$$

Equate these two infinite series directly with Euler's expression.

$\cos\theta = 1 - \theta^2/2! + \theta^4/4! \ldots$ *(even)*

$\sin\theta = \theta - \theta^3/3! \quad \theta^5/5! \ldots$ *(odd)*

which we have derived before using Maclaurin's Theorem.

Hyperbolic Functions

The functions $\cosh x$ and $\sinh x$ are the even and odd functions of e^x.

So $\cosh x = 1 + x^2/2! + x^4/4! \ldots$

and $\sinh x = x + x^3/3! + x^5/5! \ldots$

Hyperbolic to Circular Functions

If we write $e^{ix} = \cos x + i\sin x$

and $e^x = \cosh x + \sinh x$

Hence $e^{ix} = \cosh ix + \sinh ix$

we can read off the following relationships

$\cosh ix = \cos x$ and **$\sinh ix = i\sin x$**

It can also be shown that

$\cos x = ½\,(e^{ix} + + e^{-ix})$ ❶

and $\cosh x = ½\,(e^x + e^{-x})$ ❷

Substitute ix for x in ❶.

$\cos ix = ½\,(e^{iix} + + e^{-iix})$

$= ½\,(e^{-x} + + e^{x})$ ❸\`

Now ❸= ❶ so **$\cos ix = \cosh x$**

Also $\sin x = ½_i\,(e^{ix} - e^{-ix})$

so $\sin ix = ½_i\,(e^{iix} - e^{-iix})$

$\sin ix = ½_i\,(e^{-x} - e^{x})$

Now $\sinh x = ½\,(e^x - e^{-x})$

$= -\,½\,(e^{-x} - e^{x})$

and equating so **$\sin ix = i\sinh x$**

(taking careful note of signs and i)

These relationships will take you through to 1st Year University

Robert Goodhand

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