# Propositional Calculus - Truth Table

| Logic Gate Terminology | | | contradiction | NOR | non-implication | inversion | non-implication | inversion | ~XOR / non-equivalence | NAND | AND | equivalence / XOR | amplifier | implication | amplifier | implication | OR | tautology |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | p | q | | | | | | | | | | | | | | | | |
| | 1 | 1 | F | F | F | F | F | F | F | F | T | T | T | T | T | T | T | T |
| | 1 | 0 | F | F | F | F | T | T | T | T | F | F | F | F | T | T | T | T |
| | 0 | 1 | F | F | T | T | F | F | T | T | F | F | T | T | F | F | T | T |
| | 0 | 0 | F | T | F | T | F | T | F | T | F | T | F | T | F | T | F | T |
| ***OR function*** | | | ~<p v ~p> | | ~<p v ~q> | | ~<~p v q> | | ~<p v ~q> v ~<~p v q> | | ~<~p v ~q> | | q v q | | p v p | | <p v q> | |
| | | | | ~<p v q> | | ~p | | ~q | | <~p v ~q> | | ~<~p v ~q> v ~<p v q> | | <~p v q> | | <p v ~q> | | <p v ~p> |
| ***Conditional function (1)*** | | | ~<~p ⊃ ~p> | | ~<~p ⊃ ~q> | | ~<p ⊃ q> | | <p ⊃ q> ⊃ ~<~p ⊃ ~q> | | ~<p ⊃ ~q> | | <~q ⊃ q> | | <~p ⊃ p> | | <~p ⊃ q> | |
| ***Implication rule*** | | | | ~<~p ⊃ q> | | <p ⊃ ~p> | | <q ⊃ ~q> | | <p ⊃ ~q> | | <p ⊃ ~q> ⊃ ~<~p ⊃ q> | | <p ⊃ q> | | <~p ⊃ ~q> | | <~p ⊃ ~p> |
| ***Conditional function (2)*** | | | ~<p ⊃ p> | | ~<q ⊃ p> | | ~<~q ⊃ ~p> | | <~p ⊃ ~q> ⊃ ~<p ⊃ q> | | ~<q ⊃ ~p> | | *(same)* | | *(same)* | | <~q ⊃ p> | |
| ***Transposition rule*** | | | | ~<~q ⊃ p> | | *(same)* | | *(same)* | | <q ⊃ ~p> | | <~p ⊃ q> ⊃ ~<p ⊃ ~q> | | <~q ⊃ ~p> | | <q ⊃ p> | | <p ⊃ p> |
| ***AND function*** | | | <~p & p> | | <~p & q> | | <p & ~q> | | ~<p & q> & ~<~p & ~q> | | <p & q> | | <q & q> | | <p & p> | | ~<~p & ~q> | |
| ***(de Morgan's rule)*** | | | | <~p & ~q> | | ~<p & p> | | ~<q & q> | | ~<p & q> | | ~<p & ~q> & ~<~p & q> | | ~<p & ~q> | | ~<~p & q> | | ~<~p & p> |
| ***Equivalence function*** | | | p ≡ <p ⊃ ~p> | | ~p ≡ <~p ⊃ q> | | p ≡ <p ⊃ ~q> | | <p ≡ ~q> | | p ≡ <p ⊃ q> | | q ≡ <q ⊃ q> | | p ≡ <p ⊃ p> | | ~<~p ≡ <~p ⊃ ~q>> | |
| ***(exclusive OR)*** | | | | ~p ≡ <~p ⊃ ~q> | | ~<p ≡ <p ⊃ p>> | | ~<q ≡ <q ⊃ q>> | | ~<p ≡ <p ⊃ q>> | | <p ≡ q> | | ~<p ≡ <p ⊃ ~q>> | | ~<~p ≡ <~p ⊃ q>> | | ~<~p ≡ <~p ⊃ q>> |
| ***Equivalence function*** | | | | | | | | | <p & ~q> v <~p & q> | | | <p & q> v <~p & ~q> | | | | | | |
| ***NAND function*** | | | ~<~p\|p> | | ~<~p\|q> | | ~<p\|~q> | | ~<<p\|q>\|<~p\|~q>> | | ~<p\|q> | | ~<q\|q> | | ~<p\|p> | | <~p\|~q> | |
| ***(Sheffer's rule)*** | | | | ~<~p\|~q> | | <p\|p> | | <q\|q> | | <p\|q> | | <p\|q>\|<~p\|~q> | | <p\|~q> | | <~p\|q> | | <~p\|p> |
| ***NOR function*** | | | <p ⇩ ~p> | | <p ⇩ ~q> | | <~p ⇩ q> | | <p ⇩ q> ⇩ <~p ⇩ ~q> | | <~p ⇩ ~q> | | <~q ⇩ ~q> | | <~p ⇩ ~p> | | ~<p⇩ q> | |
| ***(Joint denial)*** | | | | <p ⇩ q> | | <p ⇩ p> | | <q ⇩ q> | | ~<~p ⇩ ~q> | | ~<<p ⇩ q> ⇩ <~p ⇩ ~q>> | | ~<~p ⇩ q> | | ~<p ⇩ ~q> | | ~<p ⇩ ~p> |

## Symbols of Propositional Calculus

| Symbol | Term | 1st. Meaning | | 2nd. Meaning | Logic Gate |
|---|---|---|---|---|---|
| ~ | negation | not | | | not |
| & | conjunction | and | | | and |
| v | disjunction | or | *(from Latin vel)* | | (inclusive) or |
| \| | incompatible | at least one is false | | cannot both be true | nand |
| ⇩ | joint denial | both false | | neither is true | nor |
| ≡ | biconditional | if and only if | | both true or both false | exclusive or |
| ⊃ | conditional | → if-then | | p implies q | implication |

## Axioms

| | |
|---|---|
| <p v p> ⊃ p | 1st axiom of propositional calculus |
| p ⊃ <p v q> | 2nd axiom of propositional calculus |
| <p v q> ⊃ <q v p> | 3rd axiom of propositional calculus |
| <p ⊃ q> ⊃ <<r v p> ⊃ <r v q>> | 4th axiom of propositional calculus |
| *and worthy of special mention* | |
| <p v ~p> | *Tertium-non-datur* - the excluded middle even if p is indeterminate . |

## Rules of Propositional Calculus

| | |
|---|---|
| ***Joining Rule*** | If p and q are theorems then <p & q> is a theorem. |
| ***Separation Rule*** | If <p & q> is a theorem then both p and q are theorems |
| ***Double tilde rule*** | ~~ can be deleted or inserted into any theorem *(providing it is well-formed)* |
| ***Deduction Rule*** | If q can be derived when p is assumed then <p ⊃ q> is a theorem (*a fantasy*). |
| ***Detachment Rule*** | If p and <p ⊃ q> are both theorems, then q is a theorem - *the modus ponens* |
| ***(or Derivation Rule)*** | *Deduction and Detachment are complementary* |
| ***Carry-over Rule*** | Inside a fantasy, any theorem from the reality one level higher can be used. |
| ***Implication Rule*** | <p v q> and <~p ⊃ q> are interchangeable.(Hofstadter terms the Switcheroo) |
| ***Transposition Rule*** | <p ⊃ q> and <~q ⊃ ~p> are interchangeable.(Hofstadter terms Contrapositive) |
| ***de Morgan's Rule*** | <~p & ~q> and ~<p v q> are interchangeable |

## Proofs

| | | |
|---|---|---|
| <<p ⊃ q> & <q ⊃ r>> ⊃ (p ⊃ r) | Proof by hypothetical syllogism | (if p implies q and q implies r then p implies r) |
| <<p v q> & ~q > ⊃ p | Proof by disjunctive syllogism | (If either p or q true but q false then p must be true) |
| <p v q> & <p v ~q> = p | *This is an expanded version of proof by disjunctive syllogism* | |
| <p & <p ⊃ q>> ⊃ q | Proof by indirect reasoning *Modus ponens* | (if p then q but p is true so q must be true) |
| <<p ⊃ q> & ~q> ⊃ ~p | Proof by detachment *Modus tollens* | (if p then q but if q is false then so must p) |
| <<~p ⊃ q> & <~p ⊃ ~q>> ⊃ p | Proof by contradiction *Reductio ad absurdum* . | (If ~p leads to a contradiction, p must be true) |
| <<p v q> & <<p ⊃ r> & <q ⊃ r>> ⊃ r | Proof by cases. | (if either p or q is true and either imply r, then r must also be true) |

## References

**Forever Undecided** - a puzzle guide to Godel : Raymond Smullyan : *Oxford Press*
**Mathematical Logic** : R.L. Goodstein : *Leicester University Press*
**Godel, Escher, Bach** - an eternal golden braid : Douglas R. Hofstadter : *Penguin Books*
**University of Bath Sequential Systems Tutor Notes from 1972**

## Equivalences

| | | |
|---|---|---|
| Tautologies | p = p & p | p = p v p |
| Commutative | p & q = q & p | p v q = q v p |
| Associative | p & ( q & r ) = (p & q ) & r | p v ( q v r ) = (p v q ) v r |
| Distributative | p & (q v r ) = p & q v p & r | p v (q & r ) = p v q & p v r |
| | q v ~q can be deleted from any expansion | |

***Compiled by Robert Goodhand***