

Mr. G's Handbook of Mathematics

Vectors and Matrices

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Vectors I - Linear Transformations

A transformation is linear if

$$T(a_1 \underline{v}_1 + a_2 \underline{v}_2) = a_1 T(\underline{v}_1) + a_2 T(\underline{v}_2)$$

where $T:\underline{v} \rightarrow T(\underline{v})$

The set of vectors being transformed are termed the **domain** and the new set the **image**.

Inverse Transformations

If T is a one-to-one transform then the inverse linear transform T^{-1} exists.

$$If T T^{-1} = T^{-1} T = I$$

Laws of Vector Algebra

Commutative Law

$$\underline{v} + \underline{w} = \underline{w} + \underline{v} \quad \text{Additive}$$

Associative Law

$$\underline{v} + (\underline{w} + \underline{x}) = (\underline{v} + \underline{w}) + \underline{x}$$

$$m(n\underline{v}) = mn(\underline{v})$$

$$= n(m\underline{v})$$

Multiplicative

Distributive Law

$$(m+n)\underline{v} = m\underline{v} + n\underline{v} \quad \text{Additive}$$

$$m(\underline{v} + \underline{w}) = m\underline{v} + m\underline{w} \quad \text{Multiplicative}$$

Notes

Neither vectors nor matrices obey the product commutative law.

Operations that do obey are more the exception than the rule.

eg The "son of Harry's mother" is a different person to the "mother of Harry's son".

Vectors 2 Magnitude and Modulus

Define the vector $\underline{v}$ by the matrix $\begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix}$

$$|\underline{v}| = \sqrt{(v_1^2 + v_2^2 + v_3^2)}$$

$$|\underline{v} + \underline{w}|^2 = |\underline{v}|^2 + |\underline{w}|^2$$

norm. (unit) vector or "versor"

$$\hat{u} = \underline{u} / |\underline{u}|$$

pronounced "u hat"

Unit Vectors

$$\underline{v} = x\underline{i} + y\underline{j} + z\underline{k}$$

$$\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = \underline{k} \cdot \underline{k} = 1$$

from scalar product rule

$$\underline{i} \cdot \underline{j} = \underline{k} \text{ and } \underline{j} \cdot \underline{k} = \underline{i} \text{ and } \underline{k} \cdot \underline{i} = \underline{j}$$

from scalar product rule

Hence by multiplying out $\underline{a} \cdot \underline{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

Scalar (dot •) Product

$$\underline{v} \cdot \underline{w} = |\underline{v}| |\underline{w}| \cos \theta \quad \text{this is a scalar quantity}$$

by x out in unit vector form

$$\underline{v} \cdot \underline{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$$

$$\cos \theta = (v_1 w_1 + v_2 w_2 + v_3 w_3) / (|\underline{v}| |\underline{w}|)$$

if v and w parallel

$$\underline{v} \cdot \underline{w} = x\underline{i} + y\underline{j} + z\underline{k}$$

if v and w perpendicular

$$\underline{v} \cdot \underline{w} = 0$$

resolved part $\underline{v}$ in direction $\underline{w} = \underline{v} \cdot \underline{w} / |\underline{w}|$

Vector (cross) Product

$$\underline{v} \times \underline{w} = |\underline{v}| |\underline{w}| \sin \theta \hat{u} \quad \text{this is a vector quantity}$$

$\hat{u}$ is the unit vector perpendicular to both $\underline{v}$ and $\underline{w}$ (right hand corkscrew rule)

hence $\hat{u} = \underline{u} / |\underline{u}|$

$$|\underline{v} \times \underline{w}| = |\underline{v}| |\underline{w}| \sin \theta \quad \text{this is a scalar quantity}$$

$$\underline{v} \times \underline{w} = \underline{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \underline{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \underline{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}$$

cross products are not commutative: $\underline{v} \times \underline{w} = -\underline{w} \times \underline{v}$

the vector product gives a vector perpendicular to $\underline{v}$ and $\underline{w}$ obeying the right hand rule

as an example of application $A_\Delta = \frac{1}{2} |\underline{v} \times \underline{w}|$

Vectors 3 - Equation Line

A point with position vector

$$\underline{\mathbf{a}} = a_1 \underline{\mathbf{i}} + a_2 \underline{\mathbf{j}} + a_3 \underline{\mathbf{k}}$$

and let direction vector be

$$\underline{\mathbf{v}} = v_1 \underline{\mathbf{i}} + v_2 \underline{\mathbf{j}} + v_3 \underline{\mathbf{k}}$$

$$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$$

$$\text{where } \underline{\mathbf{b}} = \underline{\mathbf{v}} - \underline{\mathbf{a}}$$

Parametric Form

$$x = x_0 + \lambda \ell$$

$$y = y_0 + \lambda m$$

$$z = z_0 + \lambda n$$

Cartesian Equation

$$(x - x_0) / \ell = (y - y_0) / m = (z - z_0) / n$$

Vector Equation Plane

$\underline{\mathbf{a}}$ is any point on the plane

$$\underline{\mathbf{r}} = \underline{\mathbf{a}} + \lambda \underline{\mathbf{u}} + \mu \underline{\mathbf{v}} \quad \text{not a lot of use}$$

$$\text{let } \underline{\mathbf{n}} = \underline{\mathbf{u}} \times \underline{\mathbf{v}} \quad \text{be a normal to the plane}$$

$$\underline{\mathbf{r}} \cdot \underline{\mathbf{n}} = \underline{\mathbf{a}} \cdot \underline{\mathbf{n}}$$

$\underline{\mathbf{r}} - \underline{\mathbf{a}}$ is a vector in the plane

and the two are perpendicular

Cartesian Equation Plane

$$ax + by + cz + d = 0$$

$$d = -\underline{\mathbf{a}} \cdot \underline{\mathbf{n}}$$

Notes

Determinants

Value associated with square matrix that determines if a set of linear equations have a solution.

The determinant of square matrix $\mathbf{A}$ is written $|\mathbf{A}|$

The determinant of the Identity matrix $|\mathbf{I}_n| = 1$ (one)

The determinant of the transpose matrix $|\mathbf{A}^T| = |\mathbf{A}|$ (the original matrix)

The determinant of the inverse matrix $|\mathbf{A}^{-1}| = 1/|\mathbf{A}|$

$|\mathbf{AB}| = |\mathbf{A}| |\mathbf{B}|$ (determinants are distributive in multiplication)

$|c\mathbf{A}| = c^n |\mathbf{A}|$ (for an $n \times n$ matrix)

$|\bar{\mathbf{A}}| = \overline{|\mathbf{A}|}$ (the det. of a complex conjugate = complex conjugate of the det..)

The **determinant** of square matrix **det A** or $|\mathbf{A}|$ is a scale factor under linear transform.

$$|\mathbf{A}| = ad - bc \quad \text{for } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A **minor** M_{ij} of matrix $\mathbf{A}$ is determinant of sub matrix after removing i^{th} row & j^{th} column.

Notes on Determinants (quick fixes to calculate the determinant)

Switching two rows or two columns changes the sign of the determinant.

Scalars can be factored out.

Adding or subtracting a row (or column) to another row (or column) doesn't change determinant.

Hence multiples of rows (or columns) can be added to other rows (or columns).

Swapping row with columns doesn't change the determinant.

A determinant with a zero row or column has value 0.

If any two rows (or columns) are identical or in proportion the determinant is zero.

(this indicates we have an **underspecified system**, so inverse matrix does not exist)

Matrices

A matrix is an array of numbers with n rows and m columns.

$$\mathbf{A}^T \cdot \mathbf{B} = \mathbf{B}^T \cdot \mathbf{A} \quad \mathbf{A} \text{ and } \mathbf{B} \text{ must be of the same order}$$

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

$$\mathbf{A}_{(n \times m)} \times \mathbf{B}_{(m \times p)} = \mathbf{C}_{(n \times p)} \quad \text{rows} \times \text{columns}$$

$$\mathbf{A} \times \mathbf{B} \neq \mathbf{B} \times \mathbf{A} \quad \text{assuming the multiplication actually exists.}$$

$$\text{A half of } \mathbf{A} = \frac{1}{2}\mathbf{A} \quad \text{nb. } \frac{1}{2} \text{ is undefined while } \frac{1}{2}\mathbf{A} \text{ is a scalar multiplication.}$$

The identity matrix is termed $\mathbf{I}$. It is a square matrix with diagonal "1"s.

$$\mathbf{A} \times \mathbf{I} = \mathbf{A}$$

$$\mathbf{I} \times \mathbf{A} = \mathbf{A}$$

$\mathbf{I}$ has 7 colleagues which rg writes $\mathbf{I}_{\text{ref.x}} \mathbf{I}_{\text{ref.y}} \mathbf{I}_{\text{rot90}} \mathbf{I}_{\text{rot-90}} \mathbf{I}_{\text{rot180}} \mathbf{I}_{\text{ref.y=x}} \mathbf{I}_{\text{ref.y=-x}}$

The **determinant** of square matrix $\det \mathbf{A}$ or $|\mathbf{A}|$ is a scale factor under linear transform.

$$\det \mathbf{A} = ad - bc$$

$$\text{for } \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

A **minor** M_{ij} of matrix $\mathbf{A}$ is determinant of sub matrix after removing i^{th} row & j^{th} column.

A **cofactor** entry C_{ij} is a "signed minor". A matrix of cofactors is termed the **conjugate**

The **transpose** of a matrix $\mathbf{A}^T$ is the matrix formed where $\mathbf{A}_{ij}$ becomes $\mathbf{A}_{ji}$

$$(\mathbf{A} \times \mathbf{B})^T = \mathbf{B}^T \times \mathbf{A}^T \quad \text{Note the exchange of } \mathbf{A} \text{ and } \mathbf{B}$$

The **transpose** of a matrix of cofactors is termed the **adjugate**.

The adjugate matrix divided by scalar quantity $\det \mathbf{A}$ gives the inverse $\mathbf{A}^{-1}$

$$\mathbf{A} \times \mathbf{A}^{-1} = \mathbf{I}$$

$\mathbf{A}^{-1}$ is the inverse of $\mathbf{A}$ which must be square matrix.

$$(\mathbf{A} \times \mathbf{B})^{-1} = \mathbf{B}^{-1} \times \mathbf{A}^{-1} \quad \text{Note the exchange of } \mathbf{A} \text{ and } \mathbf{B}$$

The inverse exists if $\det \mathbf{A}$ is non singular $\det \mathbf{A} \neq 0$

Assume $\mathbf{B}$ is unknown and $\mathbf{A}$ and $\mathbf{C}$ known where $\mathbf{A} \times \mathbf{B} = \mathbf{C}$

$$\text{then } \mathbf{B} = \mathbf{A}^{-1} \times \mathbf{C} \quad \text{and rhs can be fully determined.}$$

A matrix can have complex elements.

The conjugate transpose of such a matrix is termed the adjoint $\mathbf{A}^*$ or $\mathbf{A}^\dagger = (\bar{\mathbf{A}})^T = \bar{\mathbf{A}}^T$.

Notes on Determinants (quick fixes to calculate the determinant)

Adding or subtracting row or column to another row or column doesn't change determinant.

Swapping row with columns doesn't change the determinant.

If any two rows or columns are identical or in proportion the determinant is zero.

(this indicates we have an **underspecified system**, so inverse matrix does not exist)

Matrix Multiplication

	A			B				A x B	Transform
				point p	point q				
I	1	0	×	1	2	x coord	=	1 2	Identity
	0	1		3	4	y coord		3 4	(enlargement)
I_{ref.y}	-1	0	×	1	2	x coord	=	-1 -2	Reflection
	0	1		3	4	y coord		3 4	in y-axis
I_{ref.x}	1	0	×	1	2	x coord	=	1 2	Reflection
	0	-1		3	4	y coord		-3 -4	in x-axis
I_{rot.180}	-1	0	×	1	2	x coord	=	-1 -2	180° rotation
	0	-1		3	4	y coord		-3 -4	enlarge by -1
I_{ref.y = x}	0	1	×	1	2	x coord	=	3 4	Reflection
	1	0		3	4	y coord		1 2	in line y = x
I_{rot.-90}	0	-1	×	1	2	x coord	=	-3 -4	90° rotation
	1	0		3	4	y coord		1 2	anti-clockwise
I_{rot.90}	0	1	×	1	2	x coord	=	3 4	90° rotation
	-1	0		3	4	y coord		-1 -2	clockwise
I_{ref.y = -x}	0	-1	×	1	2	x coord	=	-3 -4	Reflection
	-1	0		3	4	y coord		-1 -2	in line y = -x

Notes

Note transforming matrix × the matrix to be transformed - "other way round" isn't consistent.

The order of operations for functions and transformations is always right to left.

Transformations

let $v \rightarrow Mv$ where v is the appropriate column matrix

Rotation through θ about origin

$$M = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{and from this we immediately derive}$$

$$\begin{bmatrix} \cos(\theta+\phi) & -\sin(\theta+\phi) \\ \sin(\theta+\phi) & \cos(\theta+\phi) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \times \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

and multiplying out the two matrices will give us the circular addition formulae

Reflection in $y = x \tan \theta$

$$M = \begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Representation Complex Numbers by Matrices

let $z = a + ib$ be represented by $a \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + b \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{\dagger}$

$$\begin{aligned} \text{Hence } z &= a + ib \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{\ddagger} \end{aligned}$$

$$\begin{aligned} \text{So } z^* &= a - ib \\ &= \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ &= \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^T \end{aligned} \quad \text{hence conjugate represented by the transpose matrix}$$

$$\begin{aligned} \text{Further } |z|^2 &= a^2 + b^2 \\ &= \begin{vmatrix} a & -b \\ b & a \end{vmatrix} \end{aligned}$$

so the modulus 2 is represented by the determinant

$$\begin{aligned} \text{Finally we can calculate } \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} &= \frac{1}{a^2+b^2} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \\ &= \frac{a - ib}{a^2 + b^2} \\ &= z^{-1} \text{ as expected} \end{aligned}$$

Notes

† These are the matrices I and $I_{\text{rot}-90}$

‡ The Jacobian matrix satisfying "Cauchy-Riemann" takes this form for mappings $P^2 \rightarrow P^2$.

The elements of matrices can be complex numbers or even matrices.

Practical Example of Matrices

Quantity Matrix

Bread Milk Eggs

2 4 3

Price Matrix

Tesco

Tesco

Bread 120

Total bill 1080

Milk 75

Eggs 180

Quantity Matrix

Bread Milk Eggs

2 4 3

Price Matrix

Tesco Asda

Tesco Asda

Bread 120 110

Total bill 1080 1070

Milk 75 70

Eggs 180 190

Quantity Matrix

Bread Milk Eggs

2 4 3

Price Matrix

Tesco Asda

Tesco Asda

Bread 120 110

1080 1070 Charlie

Milk 75 70

1410 1420 Nicole

Eggs 180 190

But if they choose a combined shop it doesn't matter which.

2490 2490

Notes

Transformations - Combined Transformations

If point P , coord (P_x, P_y) transforms to $Q (Q_x, Q_y)$ then the matrix relationship is

$\mathbf{T} \times \mathbf{P} = \mathbf{Q}$ where $\mathbf{T}$ is a 2×2 matrix, and $\mathbf{P}$ and $\mathbf{Q}$ are 2×1 matrices (the coordinates)

Such **affine** transformations (preserve straight lines and areas) can be extended to $P Q R \dots$

As all rigid body movements are rotations, translations or a combination of the two.

and setting aside a detailed matrix analysis, there are just two restrictions to consider

1) Chiral shapes are not identical to their mirror images.

2) Translations cannot be represented by a 2×2 matrix

These restrict which combined transformations can be represented by a single transform.

In general $\mathbf{T}_1 \times \mathbf{T}_2 \neq \mathbf{T}_2 \times \mathbf{T}_1$ unless otherwise stated.

A Mapping of Combined Transforms (rotation, reflection, translation)

Studying three transforms gives 9 combinations plus 2 more covering ambiguous outcomes.

1st Transform followed by **2nd Transform** equivalent to **Single Transform**

Rotation + Rotation $\equiv$ Rotation

where centre Rot_s = mean of centres Rot_1 and Rot_2 and $Rot_s \theta = Rot_1 \theta + Rot_2 \theta$
order irrelevant.

Rotation + Rotation $\equiv$ Translation

where $Rot_1 \theta + Rot_2 \theta = 0$ (no net rotation)

Rotation + Translation $\equiv$ Rotation

Translation + Rotation $\equiv$ Rotation

standard construction identifies centre rotation two similar chiral shapes. [†]

Rotation + Reflection $\equiv$ Reflection

Reflection + Rotation $\equiv$ Reflection

Reflection + Reflection $\equiv$ Rotation

if and only if the centre of rotation is at the intersect of the two lines of reflection. [‡]

The order of the transforms affects the degree of rotation.

Translation + Translation $\equiv$ Translation

Fairly obviously the combined vector translation is the vector sum of the two. Order irrelevant.

Reflection + Reflection $\equiv$ Translation

where the reflection lines are parallel (centre of rotation at infinity)

Translation + Reflection $\equiv$ does not exist

Reflection + Translation $\equiv$ does not exist

because one reflection changes the chirality of the object. Reflections must be "undone".

Notes

[†] Although $\mathbf{T}_1 \times \mathbf{T}_2 \neq \mathbf{T}_2 \times \mathbf{T}_1$ the final vector inverts and there is a sign change.

[‡] This is a key result because students are often asked to find the "second" transform.

Counting

No.	Greek	Latin
1	mono	uni
2	duo	bi
3	tri	tri
4	tetra	quad
5	penta	quin
6	hexa	sex
7	hepta	sept
8	octo	oct
9	nona	non
10	deca	dec

The "Handbook of Mathematics" has been written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. Sections of the Handbook are continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com