

Mr G's Little Book on

Differentiating

and Integrating

24 Trigonometric

Functions

*This booklet should contain an insert
showing the domains of all 72 expressions*

Differentiating Circular Functions

1) $\sin \theta$

$$\begin{aligned} & \frac{d}{d\theta} \sin \theta \\ &= \{ \sin(\theta + \delta\theta) - \sin \theta \} / \delta\theta \\ &= \{ \sin\theta \cdot \cos\delta\theta + \cos\theta \cdot \sin\delta\theta - \sin\theta \} / \delta\theta \\ &\text{as } \delta\theta \rightarrow 0 \sin \delta\theta \rightarrow \delta\theta \text{ and } \cos \delta\theta \rightarrow 1 \\ &= \cos \theta \cdot \delta\theta / \delta\theta \\ &= \cos \theta \end{aligned}$$

2) $\cos \theta$

$$\begin{aligned} & \frac{d}{d\theta} \cos \theta \\ &= \{ \cos(\theta + \delta\theta) - \cos \theta \} / \delta\theta \\ &= \{ \cos\theta \cdot \cos\delta\theta - \sin\theta \cdot \sin\delta\theta - \cos\theta \} / \delta\theta \\ &= -\sin \theta \cdot \delta\theta / \delta\theta \\ &= -\sin \theta \end{aligned}$$

3) $\tan \theta$

$$\begin{aligned} & \frac{d}{d\theta} \tan \theta \\ &= \{ \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \delta\theta \} / \cos^2 \theta \\ &= \sec^2 \theta \text{ or } 1 + \tan^2 \theta \end{aligned}$$

4) $\operatorname{cosec} \theta$

$$\begin{aligned} & \frac{d}{d\theta} \operatorname{cosec} \theta \\ &= \frac{d}{d\theta} 1 / \sin \theta \\ &= -\cos \theta / \sin^2 \theta \\ &= -\operatorname{cosec} \theta \cdot \cot \theta \end{aligned}$$

5) $\sec \theta$

$$\begin{aligned} & \frac{d}{d\theta} \sec \theta \\ &= \frac{d}{d\theta} 1 / \cos \theta \\ &= \sin \theta / \cos^2 \theta \\ &= \sec \theta \cdot \tan \theta \end{aligned}$$

6) $\cot \theta$

$$\begin{aligned} & \frac{d}{d\theta} \cot \theta \\ &= \frac{d}{d\theta} \cos \theta / \sin \theta \\ &= \{ -\sin\theta \cdot \sin\theta - \cos\theta \cdot \cos\theta \} / \sin^2 \theta \\ &= -\operatorname{cosec}^2 \theta \end{aligned}$$

7) $\sin^{-1} x$

$$\text{Let } y = \sin^{-1} x$$

$$\text{So } x = \sin y$$

$$\begin{aligned} \therefore \frac{dx}{dy} &= \cos y \\ &= \sqrt{1 - \sin^2 y} \end{aligned}$$

$$\therefore \frac{dy}{dx} = 1 / \sqrt{1 - x^2}$$

8) $\cos^{-1} x$

$$\text{Let } y = \cos^{-1} x$$

$$\text{So } x = \cos y$$

$$\begin{aligned} \therefore \frac{dx}{dy} &= -\sin y \\ &= -\sqrt{1 - \cos^2 y} \end{aligned}$$

$$\therefore \frac{dy}{dx} = -1 / \sqrt{1 - x^2}$$

Differentiating Hyperbolics

9) $\tan^{-1} x$

$$\text{Let } y = \tan^{-1} x$$

$$\text{So } x = \tan y$$

$$\therefore \frac{dx}{dy} = 1 + \tan^2 y \quad [\text{from 3}]$$

$$= 1 + x^2$$

$$\therefore \frac{dy}{dx} = 1 / (1 + x^2)$$

10) $\operatorname{cosec}^{-1} x$

$$\text{Let } y = \operatorname{cosec}^{-1} x$$

$$\text{So } x = \operatorname{cosec} y$$

$$\therefore \frac{dx}{dy} = -\operatorname{cosec} y \cdot \cot y \quad [\text{from 4}]$$

$$\text{now } \operatorname{cosec}^2 y = 1 + \cot^2 y$$

$$\text{So } \frac{dy}{dx} = -1 / \{ |x| \sqrt{x^2 - 1} \}$$

11) $\sec^{-1} x$

$$\text{Let } y = \sec^{-1} x$$

$$\text{So } x = \sec y$$

$$\therefore \frac{dx}{dy} = \sec y \cdot \tan y \quad [\text{from 5}]$$

$$\text{now } \sec^2 y = 1 + \tan^2 y$$

$$\text{So } \frac{dy}{dx} = 1 / \{ |x| \sqrt{x^2 - 1} \}$$

12) $\cot^{-1} x$

$$\text{Let } y = \cot^{-1} x$$

$$\text{So } x = \cot y$$

$$\therefore \frac{dx}{dy} = -\operatorname{cosec}^2 y. \quad [\text{from 6}]$$

$$= -(1 + \cot^2 y)$$

$$\text{So } \frac{dy}{dx} = -1 / (x^2 + 1)$$

13) $\sinh x$

$$\sinh x = x + x^3 / 3! + x^5 / 5! + x^7 / 7! \dots$$

$$\text{So } \frac{d \sinh x}{dx} = 1 + x^2 / 2! + x^4 / 4! \dots$$

$$= \cosh x$$

14) $\cosh x$

$$\cosh x = 1 + x^2 / 2! + x^4 / 4! + x^6 / 6! \dots$$

$$\text{So } \frac{d \cosh x}{dx} = x + x^3 / 3! + x^5 / 5! \dots$$

$$= \sinh x$$

But that is not particularly enlightening so

let's try an alternative way

$$\cosh x = \frac{1}{2} (e^x + e^{-x})$$

$$\frac{d \cosh x}{dx} = \frac{1}{2} (e^x - e^{-x})$$

$$= \sinh x$$

15) $\tanh x$

$$\cosh x = 1 + x^2 / 2! + x^4 / 4! + x^6 / 6! \dots$$

$$\frac{d \tanh x}{dx} = \frac{d}{dx} (\sinh x / \cosh x)$$

$$= (\cosh x \cdot \cosh x - \sinh x \cdot \sinh x) / \cosh^2 x$$

$$\text{remembering } \cosh^2 x - \sinh^2 x = 1$$

$$\text{hence } 1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$= 1 - \tanh^2 x$$

$$= \operatorname{sech}^2 x$$

16) $\operatorname{cosech} x$

$$\frac{d \operatorname{cosech} x}{dx} = \frac{d}{dx} (1 / \sinh x)$$

$$= -\cosh x / \sinh^2 x$$

$$= -\operatorname{cosech} x \cdot \cot x$$

17) sech x

$$\frac{d \operatorname{sech} x}{dx} = \frac{d}{dx} \left(1 / \cosh x \right)$$

$$= - \sinh x / \cosh^2 x$$

$$= - \operatorname{sech} x \cdot \tanh x$$

18) coth x

$$\frac{d \coth x}{dx} = \frac{d}{dx} \left(\cosh x / \sinh x \right)$$

$$= (\sinh x \cdot \sinh x - \cosh x \cdot \cosh x) / \sinh^2 x$$

$$= 1 - \coth x$$

$$= - \operatorname{cosech}^2 x$$

19) sinh⁻¹ x

$$\text{Let } y = \sinh^{-1} x$$

$$\text{So } x = \sinh y$$

$$\therefore \frac{dx}{dy} = \cosh y$$

$$= \sqrt{x^2 + 1}$$

$$\therefore \frac{dy}{dx} = 1 / \sqrt{x^2 + 1}$$

20) cosh⁻¹ x

$$\text{Let } y = \cosh^{-1} x$$

$$\text{So } x = \cosh y$$

$$\therefore \frac{dx}{dy} = \sinh y$$

$$= \sqrt{x^2 - 1}$$

$$\therefore \frac{dy}{dx} = 1 / \sqrt{x^2 - 1}$$

21) tanh⁻¹ x

$$\text{Let } y = \tanh^{-1} x$$

$$\text{So } x = \tanh y$$

$$\therefore \frac{dx}{dy} = \operatorname{sech}^2 y \quad [from 15]$$

$$= (1 - \tanh^2 y)$$

$$\therefore \frac{dy}{dx} = 1 / (1 - x^2)$$

22) cosech⁻¹ x

$$\text{Let } y = \operatorname{cosech}^{-1} x$$

$$\text{So } x = \operatorname{cosech} y$$

$$\therefore \frac{dx}{dy} = - \operatorname{cosech} y \cdot \coth y \quad [from 4]$$

$$\text{now } \operatorname{cosech}^2 y = \coth^2 y - 1$$

$$\therefore \frac{dy}{dx} = -1 / \{ x \sqrt{1 + x^2} \}$$

23) sech⁻¹ x

$$\text{Let } y = \operatorname{sech}^{-1} x$$

$$\text{So } x = \operatorname{sech} y$$

$$\therefore \frac{dx}{dy} = - \operatorname{sech} y \cdot \tanh y \quad [from 17]$$

$$\text{now } \operatorname{sech}^2 y = 1 - \tanh^2 y$$

$$\therefore \frac{dy}{dx} = 1 / \{ x \sqrt{1 - x^2} \}$$

24) coth⁻¹ x

$$\text{Let } y = \coth^{-1} x$$

$$\therefore x = \coth y$$

$$\therefore \frac{dx}{dy} = - \operatorname{cosech}^2 y. \quad [from 18]$$

$$= 1 - \coth^2 y$$

$$\therefore \frac{dy}{dx} = 1 / (1 - x^2)$$

Integrating Circular Functions

25) $\sin \theta$

$$\int \sin \theta \, d\theta = -\cos \theta + c$$

26) $\cos \theta \, d\theta$

$$\int \cos \theta \, d\theta = \sin \theta + c$$

27) $\tan \theta$

$$\int \tan \theta \, d\theta = \int (\sin \theta / \cos \theta) \, d\theta$$

Now from the log rule of differentiation

$$\frac{d}{dx} [\ln f(x)] = f'(x) / f(x)$$

$$\begin{aligned} \rightarrow &= -\ln |\cos \theta| + c \\ &= \ln |\sec \theta| + c \end{aligned}$$

28) $\operatorname{cosec} \theta$

$$\text{let } t = \tan \frac{1}{2} \theta$$

$$\frac{dt}{d\theta} = \frac{1}{2} (1 + t^2) \quad [\text{from 3}]$$

$$d\theta = \left\{ \frac{2}{1 + t^2} \right\} dt$$

Now here's the clever bit

$$\text{because } \sec^2 \theta = 1 + \tan^2 \theta$$

$$\text{then } \sin 2\theta = (2 \tan \theta) / (1 + \tan^2 \theta)$$

$$\text{So } \sin \theta = \left\{ \frac{2t}{1 + t^2} \right\}$$

Putting this all together

$$\int \operatorname{cosec} \theta \, d\theta$$

$$= \int \left\{ \frac{(1 + t^2)}{2t} \right\} \times \left\{ \frac{2}{1 + t^2} \right\} dt$$

$$= \int 1/t \, dt$$

$$= \ln (\tan \frac{1}{2} \theta)$$

$$= \ln |(\operatorname{cosec} \theta - \cot \theta)|$$

$$- \ln |(\operatorname{cosec} \theta + \cot \theta)| + c$$

29) $\sec \theta$

$$\int \sec \theta \, d\theta = \int 1 / \cos \theta \, d\theta$$

$$= \int (\cos \theta / \cos^2 \theta) \, d\theta$$

$$= \int (\cos \theta / 1 - \sin^2 \theta) \, d\theta$$

$$\text{Let } u = \sin \theta \text{ then } du = \cos \theta \, d\theta$$

and our integral becomes

$$\int du / (1 - u^2) = \tanh^{-1} u \quad [\text{from 21}]$$

$$\therefore \int \sec \theta \, d\theta = \tanh^{-1} (\sin \theta) + c$$

The solution given in text books is more usually $\ln |\sec \theta + \tan \theta|$ but these are equivalent

30) $\cot \theta$

$$\int \cot \theta \, d\theta = \int (\cos \theta / \sin \theta) \, d\theta$$

which again is a standard form giving

$$= \ln |(\sin \theta)| + c$$

31) $\sin^{-1} x$

$$\int \sin^{-1} x \, dx = \int 1 \times \sin^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \sin^{-1} x - \int \{ x / \sqrt{1-x^2} \} \\ = x \sin^{-1} x + \sqrt{1-x^2} + c \end{aligned}$$

32) $\cos^{-1} x \, dx$

$$\int \cos^{-1} x \, dx = \int 1 \times \cos^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \cos^{-1} x - \int \{ -x / \sqrt{1-x^2} \} \, dx \\ = x \cos^{-1} x - \sqrt{1-x^2} + c \end{aligned}$$

Integrating Hyperbolics

33) $\tan^{-1} x \, dx$

$$\int \tan^{-1} x \, dx = \int 1 \times \tan^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \tan^{-1} x - \int \{ x / \sqrt{1 + x^2} \} \, dx \\ = x \tan^{-1} x - \frac{1}{2} \ln (x^2 + 1) + c \end{aligned}$$

34) $\operatorname{cosec}^{-1} x \, dx$

$$\int \operatorname{cosec}^{-1} x \, dx = \int 1 \times \operatorname{cosec}^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \operatorname{cosec}^{-1} x - \int \{ -x / x \sqrt{x^2 - 1} \} \, dx \\ = x \operatorname{cosec}^{-1} x + \cosh^{-1} |x| + c \end{aligned}$$

35) $\sec^{-1} x \, dx$

$$\int \sec^{-1} x \, dx = \int 1 \times \sec^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \sec^{-1} x - \int x / \{ x \sqrt{x^2 - 1} \} \, dx \\ = x \sec^{-1} x - \cosh^{-1} x + c \end{aligned}$$

36) $\cot^{-1} x \, dx$

$$\int \cot^{-1} x \, dx = \int 1 \times \cot^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \cot^{-1} x - \int \{ -x / (1 + x^2) \} \, dx \\ = x \cot^{-1} x + \frac{1}{2} \ln (x^2 + 1) + c \end{aligned}$$

37) $\sinh x \, dx$

$$\int \sinh x \, dx = \cosh x + c$$

38) $\cosh x \, dx$

$$\int \cosh x \, dx = \sinh x + c$$

39) $\tanh x \, dx$

$$\begin{aligned} \int \tanh x \, dx &= \int (\sinh x / \cosh x) \, dx + c \\ &= \ln (\cosh x) + c \end{aligned}$$

40) $\operatorname{cosech} x$

Taking our cue from $\int \operatorname{cosec} x \, dx$

let $t = \tanh \frac{1}{2} x$

$$\frac{dt}{dx} = \frac{1}{2} (1 - t^2) \quad [\text{from 15}]$$

$$dx = 2 / (1 - t^2) \, dt$$

and by similar reasoning to [15]

$$\sinh x = 2t / (1 - t^2)$$

Putting this all together

$$\begin{aligned} \int \operatorname{cosech} x \, dx \\ &= \int \{ (1 - t^2) / 2t \} \times \{ 2 / (1 - t^2) \} \, dt \\ &= \int 1 / t \, dt = \ln (\tanh \frac{1}{2} x) \\ &= \ln |(\coth x - \operatorname{cosech} x)| + c \end{aligned}$$

41) sech x dx

Taking our cue from $\int \sec x \, dx$

$$\int \operatorname{sech} x \, dx = \int \left(\cosh x / \cosh^2 x \right) dx$$

$$= \int \left\{ \cosh x / (1 + \sinh^2 x) \right\} dx$$

$$\text{Let } u = \sinh x$$

$$du = \cosh x \, dx$$

$$\int du / (1 + u^2) = \tan^{-1} u \quad [\text{from 9}]$$

$$\text{so } \int \operatorname{sech} x \, dx = \tan^{-1} (\sinh x) + c$$

42) coth x

$$\int \coth x \, dx = \int (\cosh x / \sinh x) \, dx$$

$$= \ln (\sinh x) + c$$

43) $\sinh^{-1} x \, dx$

$$\int \sinh^{-1} x \, dx = \int 1 \times \sinh^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \sinh^{-1} x - \int \{x / \sqrt{x^2 + 1}\} \\ = x \sinh^{-1} x + \sqrt{x^2 + 1} + c \end{aligned}$$

44) $\cosh^{-1} x$

$$\int \cosh^{-1} x \, dx = \int 1 \times \cosh^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \cosh^{-1} x - \int x / \sqrt{x^2 - 1} \, dx \\ = x \cosh^{-1} x - \sqrt{x^2 - 1} + c \end{aligned}$$

45) $\tanh^{-1} x \, dx$

$$\int \tanh^{-1} x \, dx = \int 1 \times \tanh^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \tanh^{-1} x - \int x / \sqrt{1 - x^2} \, dx \\ = x \tanh^{-1} x - \frac{1}{2} \ln (1 - x^2) + c \end{aligned}$$

46) $\operatorname{cosech}^{-1} x$

$$\int \operatorname{cosech}^{-1} x \, dx = \int 1 \times \operatorname{cosech}^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \operatorname{cosech}^{-1} x - \int \{-x / x \sqrt{x^2 + 1}\} dx \\ = x \operatorname{cosech}^{-1} x + \sinh^{-1} x + c \end{aligned}$$

47) $\operatorname{sech}^{-1} x$

$$\int \operatorname{sech}^{-1} x \, dx = \int 1 \times \operatorname{sech}^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \operatorname{sech}^{-1} x - \int \{-x / x \sqrt{1 - x^2}\} dx \\ = x \operatorname{sech}^{-1} x - \cosh^{-1} x + c \end{aligned}$$

48) $\coth^{-1} x \, dx$

$$\int \coth^{-1} x \, dx = \int 1 \times \coth^{-1} x \, dx$$

and integrating by parts give

$$\begin{aligned} x \coth^{-1} x - \int \{x / (1 - x^2)\} dx \\ = x \coth^{-1} x + \frac{1}{2} \ln |1 - x^2| + c \end{aligned}$$

Footnote

In my first cut of this booklet I ignored all the principles of domain restriction. This did not become apparent until the last integral $\coth^{-1} x \, dx$ where I could see there was a mismatch between the domains of the integrand and integral.

I therefore mapped out the domains of all 72 expressions as shown on the insert and could see 5 domain issues. These I then corrected by inserting absolute values eg $|x|$ into specific terms.

Appendix

$\int \coth^{-1} x$

Derive gives the integral as

$$\frac{1}{2} (x - \frac{1}{2}) \ln(x+1)/\ln(x-1) - \ln(\frac{1}{x+1}).$$

$$\text{Now } x \coth^{-1} x + \frac{1}{2} \ln(x^2-1)$$

$$\begin{aligned} &= \frac{1}{2} x [\ln(x+1) - \ln(x-1)] + \\ &\quad \frac{1}{2} [\ln(x+1) + \ln(x-1)] \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} \ln(x-1) - \frac{1}{2} x \ln(x-1) + \\ &\quad \frac{1}{2} \ln(x-1) + \ln(x+1) \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} x \ln(x-1) + \frac{1}{2} \ln(x+1) + \\ &\quad \frac{1}{2} \ln(x-1) - \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} x \ln(x-1) + \ln(x+1) - \\ &\quad \frac{1}{2} \ln(x+1) + \frac{1}{2} \ln(x-1) \\ &= \frac{1}{2} x \ln(x+1) - \frac{1}{2} \ln(x+1) - \frac{1}{2} x \ln(x-1) + \\ &\quad \frac{1}{2} \ln(x-1) + \ln(x+1) \\ &= \frac{1}{2} (x-1) [\ln(x+1)] - \frac{1}{2} (x-1) [\ln(x-1)] + \\ &\quad \ln(x+1) \\ &= \frac{1}{2} (x-1) [\ln(x+1) - \ln(x-1)] + \ln(x+1) \end{aligned}$$

which matches above given last term is
strangely inverted.

*Alexa DOES give the integral as first
derived.*