The Nature of the Exponential Function

Assume there exists a function EXP(x) such that ${}^{d}/_{dx}$ EXP(x) = EXP (x) ie the function is its own derivative.

Assume it can be expressed as a series

$EXP(x) = a_0 + a_1x + a_2x^2 \ldots$

Each term must itself be the derivative of the succeeding term. So for simplicity set $a_0 = 1$.

Hence $a_1 = x$, $a_2 = {}^{x^2}/_{2!}$, $a_3 = {}^{x^3}/_{3!}$ etc. so

$EXP(x) = 1 + x + {}^{x^2}/_{2!} + {}^{x^3}/_{3!} \ldots$

setting x = 1 we have

$EXP(1) = 1 + 1 + {}^{1}/_{2!} + {}^{1}/_{3!} \ldots$

Assuming the series to be convergent we denote the sum by e so EXP(1) = e and in passing EXP(0) = 1

We need to demonstrated $EXP(x) = e^x$

The first step is to demonstrate that

EXP(a) + EXP(b) = EXP (a + b)

so we multiply out two infinite series

$(1+a+{}^{a^2}/_{2!}+{}^{a^3}/_{3!} \ldots) \times (1+b+{}^{b^2}/_{2!}+{}^{b^3}/_{3!} \ldots)$

If you can be bothered to write out all the terms in appropriate columns you will find within each factorial group the binomial terms appearing give

$1 + (a+b) + {}^{1}/_{2!}(a+b)^2 + {}^{1}/_{3!}(a+b)^3 \ldots$

Hence EXP(a) + EXP(b) = EXP (a + b)

So now we are the first step in the journey to show EXP is indeed an exponential function.

Using the same technique of multiplying out infinite series the result may be extended to

$EXP(a) \times EXP(b) \times EXP(c)$

$= EXP(a+b+c)$

Hence $[EXP(a)]^x = EXP(ax)$

Setting a = 1

$[EXP(1)]^x = EXP(x)$

but we have already set EXP(1) = e

so now we have shown *(roll of drums)*

$EXP(x) = e^x$. and further $e^x \times e^y = e^{x+y}$

This process is valid at this stage for x an integer but by similar processes we can extend into x rational, x negative and ultimately x real.

Now we extend into the complex plane.

Let $y = \cos\theta + i \sin\theta$

${}^{dy}/_{d\theta} = -\sin\theta + i\cos\theta$

${}^{dy}/_{d\theta} = iy$ and thus $\int {}^{dy}/_{y} = \int i d\theta$

$\ln y = i\theta + k$ and easily shown

at $\theta = 0$, $y = 1$ $\ln y = 0$ hence $k=0$

Hence $y = e^{i\theta}$ and hence

$e^{i\theta} = \cos\theta + i\sin\theta$

Setting $\theta = \pi$ we get $e^{i\pi} = -1$

which is the neatest proof you've ever seen

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