1.0 Peano's Axioms of Arithmetic

1) 1 is a (natural) number
2) Every number has a successor
3) 1 is not the successor of any number.
4) Two numbers whose successors are equal are also equal.
5) If something is true for an unspecified number and its successor and also for 1 then it's true for all natural numbers.

Axiom 5 is termed "proof by induction". Axiom 5 cannot itself be proved from within the system.

2.0 Proof by Induction

Prove $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n + 1)(2n + 1)$

Proof

Establish expression for $\Sigma (r + 1)^2$ by two different methods.

Method 1

$\Sigma (r + 1)^2 =$
$\frac{1}{6}n(n + 1)(2n + 1) + (n + 1)^2 =$
$\frac{1}{6}(n^2 + n)(2n + 1) + \frac{6}{6}(n + 1)^2 =$
$\frac{1}{6}(2n^3 + n^2 + 3n) + \frac{6}{6}(n^2 + 2n +1) =$
$\frac{1}{6}\{(2n^3 + n^2 + 3n) + (6n^2 + 12n +6)\}$
$= \frac{1}{6}(2n^3 + 7n^2 + 5n + 6)$

Method 2

Let $n \Rightarrow n + 1$ so $\Sigma (r + 1)^2 =$
$\frac{1}{6}(n + 1)(n + 2)(2n + 3) =$
$\frac{1}{6}(n^2 + 3n + 2)(2n + 3) =$
$\frac{1}{6}(2n^3 + 7n^2 + 5n + 6)$
which is identical to the Method 1 result. Finally we show that for $r = 1$
$\Sigma r^2 = \frac{1}{6}1(1 + 1)(2 + 1) = 1$ True.

Therefore from Axiom 5 we have proved the expression for all n.

3.0 Telescoping

The above proof leaves open the question "how was the original function derived?" Did we just try out every conceivable expression until we hit on the correct one? Summation formulae can be derived from first principles using the method of "telescoping".

Summation 1

$\sum_{r=1}^{n} k = nk$ and specifically
$\sum_{r=1}^{n} 1 = n$

This is considered self-evident although it can be proved rigorously.

Summation n

Let $S_1 = \sum_{r=1}^{n} r$

$$(n + 1)^2 - n^2 = n^2 + 2n + 1 - n^2 = n + 1$$

$$2^2 - 1^2 = 2(1) + 1 \quad \text{ie } 3$$
$$3^2 - 2^2 = 2(2) + 1 \quad \text{ie } 5$$
$$4^2 - 3^2 = 2(3) + 1 \quad \text{ie } 7$$

This is the same result we might get by adding extra squares to make the next highest square.

$$n^2 - (n - 1)^2 = 2 (n - 1) + 1$$
$$(n + 1)^2 - n^2 = 2n + 1$$

Add up all the lhs and the rhs to get

$$(n + 1)^2 - 1^2 = 2S_1 + n$$

Rearranging

$$n^2 + 2n = 2S_1 + n$$
$$S_1 = \tfrac{1}{2} n(n + 1)$$

So we have produced the summation of the first n numbers just by using the single assumption that summing 1 for n times is equal to n.

Summation n^2

We now use the formula for S_1 to find $S_2 = {}_{r=1}\Sigma^n r^2$ using the same procedure.

$$(n + 1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3$$
$$= 3n^2 + 3n + 1$$

$2^3 - 1^3 = 3(1)^2 + 3(1) + 1$ ie 7
$3^3 - 2^3 = 3(2)^2 + 3(2) + 1$ ie 19
$4^3 - 3^3 = 3(3)^2 + 3(3) + 1$ ie 37

The increases are centred hexagonal numbers – the extra cubes you have to add to get the next highest cube.

$$n^3 - (n - 1)^3 = 3(n - 1)^2 + 3(n - 1) + 1$$
$$(n + 1)^3 - n^3 = 3n^2 + 3n + 1$$

Add up all the lhs and the rhs to get
$$(n + 1)^3 - (n - 1)^3 = 3S_2 + 3S_1 + n$$

Rearranging
$$n^3 + 3n^2 + 3n = 3S_2 + 3S_1 + n$$
$$2n^3 + 6n^2 + 6n = 6S_2 + 3n^2 + 3n + 2n$$
$$6S_2 = 2n^3 + 3n^2 + n$$
$$= n (2n^2 + 3n + 1)$$
$$= n (2n + 1) (n + 1)$$
$$S_2 = {}^1/_6\, n (2n + 1) (n + 1)$$

Outline of Summation for n^3

We set up the equation
$(n + 1)^4 - n^4 = 4n^3 + 6n^2 + 4n + 1$

And use for n= 1, 2 , 3 … as before. Then we telescope the two sides to produce an expression containing S_3, S_2 and S_1. But as we already know S_2 and S_1 we rearrange to give an expression for S_3.

This procedure can be continued indefinitely.

Summary of Summations

$S_0(n) = {}^1/_1 n^1$
$S_1(n) = {}^1/_2 n^2 + \frac{1}{2} n^1$
$S_2(n) = {}^1/_3 n^3 + \frac{1}{2} n^2 + {}^1/_6 n^1$
$S_3(n) = {}^1/_4 n^4 + \frac{1}{2} n^3 + {}^1/_4 n^2 + 0n^1$
$S_4(n) = {}^1/_5 n^5 + \frac{1}{2} n^4 + {}^1/_3 n^3 + 0n^2 + {}^{-1}/_{30} n^1$
$S_5(n) =$
${}^1/_6 n^6 + \frac{1}{2} n^5 + {}^5/_{12} n^4 + 0n^3 + {}^{-1}/_{12} n^2 + 0n^1$
and a pattern of patterns starts to emerge.

The author has calculated to S_{12}. The sequence of the last term n is

$1, \frac{1}{2}, {}^1/_6, 0, {}^{-1}/_{30}, 0, {}^1/_{42}, 0, {}^{-1}/_{30}, 0, {}^5/_{66}$

These are termed Bernouilli numbers. They crop up in many mathematical relationships.

Many of the formula above can be factorised - most significantly
$S_3 = \{\frac{1}{2} n (n + 1)\}^2$
that is $S_3 = S_1^2$ which is pretty neat and worth remembering.

Telescoping

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$= n + 1$

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Rearranging
$n^3 + 3n^2 + 3n \quad = 3S_2 + 3S_1 + n$
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$S_1(n) = {}^1/_2n^2+{}^1/_2n^1$
$S_2(n) = {}^1/_3n^3+{}^1/_2n^2+{}^1/_6n^1$
$S_3(n) = {}^1/_4n^4+{}^1/_2n^3+{}^1/_4n^2+ 0n^1$
$S_4(n) = {}^1/_5n^5+{}^1/_2n^4+{}^1/_3n^3+ 0n^2 + {}^{-1}/_{30}n^1$
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