Finding Cubic Equations from Forward Difference Tables

In any forward difference table, calculate each line of differences until you get a constant. That tells you the order of the equation and the first coefficient.

One line then it's linear and the ?**n** is the last line ÷ 1

Two lines then it's a quadratic and the ?$\mathbf{n}^2$ is the last line ÷ (1x2) written 2!

Three lines then it's a cubic and the ?$\mathbf{n}^3$ is the last line ÷ (1x2x3) written 3!

Four lines then it's a quartic and the ?$\mathbf{n}^4$ is the last line ÷ (1x2x3x4) written 4!

Here's an example

0	1	2	3	4	5	6
	−4	**−4**	**10**	**50**	**128**	**256**

Construct the forward difference table and deduce the value at 0

0	1	2	3	4	5	6
−2	**−4**	**−4**	**10**	**50**	**128**	**256**
	−2	0	14	40	78	128
		2	14	26	38	50
			12	12	12	12

After three lines, the constant string is 12 so the first term is $^{12}/_{3!} = 2\mathbf{n}^3$

Now take out that $2\mathbf{n}^3$ *value from the original set of results.*

	0	1	2	3	4	5	6
Original Set Results	**−2**	**−4**	**−4**	**10**	**50**	**128**	**256**
Less $2\mathbf{n}^3$	0	2	16	54	128	250	432
Gives	−2	−6	−20	−44	−78	−122	−176

Now being very careful on signs, carry on from the beginning with these new results.

0	1	2	3	4	5	6
−2	**−6**	**−20**	**−44**	**−78**	**−122**	**−176**
−4	−14	−24	−34	−44	−54	
	−10	−10	−10	−10	−10	

Because we've taken out the $2\mathbf{n}^3$ *term, we're left with a quadratic or possibly even simpler if there is no* n^2 *term. Note the constant doesn't change either.*

We can immediately deduce the other terms. The ?$\mathbf{n}^2$ *term is half −10*

Hence $-5n^2 + ?n - 2 = -6$ at $n = 1$ hence $? = -6 + 5 + 2 = 1$

and we have the final expression $2\mathbf{n}^3 - 5\mathbf{n}^2 + \mathbf{n} - 2$.

Here is a harder example

Start with your results	**0**	**1**	**2**	**3**	**4**	**5**	**6**
		1	***2***	***4***	***9***	***19***	***36***

Construct the forward difference table and deduce the value at 0.

0	**1**	**2**	**3**	**4**	**5**	**6**
–1	***1***	***2***	***4***	***9***	***19***	***36***
	2	*1*	*2*	*5*	*10*	*17*
		–1	*1*	*3*	*5*	*7*
			2	2	2	2

The constant is 2 so the first term is ${}^{1}/_{3}\,\mathbf{n}^3$ (because $2 \div (1 \times 2 \times 3) = {}^{1}/_{3}$

Now take out that ${}^{1}/_{3}\,\mathbf{n}^3$ value from the original set of results.

	0	**1**	**2**	**3**	**4**	**5**	**6**
Original Set Results	***1***	***1***	***2***	***4***	***9***	***19***	***36***
Less ${}^{1}/_{3}\,\mathbf{n}^3$	*0*	${}^{-1}/_{3}$	${}^{-8}/_{3}$	${}^{-27}/_{3}$	${}^{-64}/_{3}$	${}^{-125}/_{3}$	${}^{-216}/_{3}$
Gives	*–1*	${}^{2}/_{3}$	${}^{-2}/_{3}$	${}^{-15}/_{3}$	${}^{-37}/_{3}$	${}^{-68}/_{3}$	${}^{-108}/_{3}$

Because we've now introduced a ${}^{1}/_{3}$ fraction, just work through the rest in thirds. Also be very careful on signs and carry on with these new results, which be a quadratic.

0	**1**	**2**	**3**	**4**	**5**	**6**
–1	$\mathbf{{}^{2}/_{3}}$	$\mathbf{{}^{-2}/_{3}}$	$\mathbf{{}^{-15}/_{3}}$	$\mathbf{{}^{-37}/_{3}}$	$\mathbf{{}^{-68}/_{3}}$	$\mathbf{{}^{-108}/_{3}}$
	${}^{5}/_{3}$	${}^{-4}/_{3}$	${}^{-13}/_{3}$	${}^{-22}/_{3}$	${}^{-31}/_{3}$	${}^{-40}/_{3}$
		–3	*–3*	*–3*	*–3*	*–3*

The $?\mathbf{n}^2$ term is half of (–3) ie $({}^{-3}/_{2}\,\mathbf{n}^2)$.

So at $n = 1$ we have ${}^{-3}/_{2} + ? - 1 = {}^{2}/_{3}$ so $? = {}^{19}/_{6}$

Put the whole lot together and we have ${}^{1}/_{3}\,\mathbf{n}^3 - {}^{3}/_{2}\,\mathbf{n}^2 + {}^{19}/_{6}\,\mathbf{n} - 1$

Actually that's not as bad as it looks because if we take out a factor ${}^{1}/_{6}$ we get

$${}^{1}/_{6}\,(2\mathbf{n}^3 - 9\mathbf{n}^2 + 19\mathbf{n} - 6)$$ (which doesn't look too bad.)

Factors of ${}^{1}/_{3}$ and ${}^{1}/_{6}$ are very common in cubics (why?)

rg

Finding Cubic Equations from Forward Difference Tables

In any forward difference table, calculate each line of differences until you get a constant. That tells you the order of the equation and the first coefficient.

One line then it's linear and the ?**n** is the last line ÷ 1

Two lines then it's a quadratic and the ?$\mathbf{n}^2$ is the last line ÷ (1x2) written 2!

Three lines then it's a cubic and the ?$\mathbf{n}^3$ is the last line ÷ (1x2x3) written 3!

Four lines then it's a quartic and the ?$\mathbf{n}^4$ is the last line ÷ (1x2x3x4) written 4!

Here's an example

0	1	2	3	4	5	6
	–4	**–4**	**10**	**50**	**128**	**256**

Construct the forward difference table and deduce the value at 0

0	1	2	3	4	5	6
–2	**–4**	**–4**	**10**	**50**	**128**	**256**
	–2	0	14	40	78	128
		2	14	26	38	50
			12	12	12	12

After three lines, the constant string is 12 so the first term is $^{12}/_{3!} = 2\mathbf{n}^3$

Now take out that $2\mathbf{n}^3$ *value from the original set of results.*

	0	1	2	3	4	5	6
Original Set Results	**–2**	**–4**	**–4**	**10**	**50**	**128**	**256**
Less $2\mathbf{n}^3$	0	2	16	54	128	250	432
Gives	–2	–6	–20	–44	–78	–122	–176

Now being very careful on signs, carry on from the beginning with these new results.

0	1	2	3	4	5	6
–2	**–6**	**–20**	**–44**	**–78**	**–122**	**–176**
–4	–14	–24	–34	–44	–54	
	–10	–10	–10	–10	–10	

Because we've taken out the $2\mathbf{n}^3$ *term, we're left with a quadratic or possibly even simpler if there is no* n^2 *term. Note the constant doesn't change either.*

We can immediately deduce the other terms. The ?$\mathbf{n}^2$ *term is half –10*

Hence $-5n^2 + ?n - 2 = -6$ at $n = 1$ hence $? = -6 + 5 + 2 = 1$

and we have the final expression $2\mathbf{n}^3 - 5\mathbf{n}^2 + \mathbf{n} - 2$.

Here is a harder example

Start with your results	**0**	**1**	**2**	**3**	**4**	**5**	**6**
		1	***2***	***4***	***9***	***19***	***36***

Construct the forward difference table and deduce the value at 0.

0	**1**	**2**	**3**	**4**	**5**	**6**
–1	***1***	***2***	***4***	***9***	***19***	***36***
	2	1	2	5	10	17
		–1	1	3	5	7
			2	2	2	2

The constant is 2 so the first term is $^1/_3\,\mathbf{n}^3$ (because $2 \div (1 \times 2 \times 3) = {}^1/_3$

Now take out that $^1/_3\,\mathbf{n}^3$ value from the original set of results.

	0	**1**	**2**	**3**	**4**	**5**	**6**
Original Set Results	***1***	***1***	***2***	***4***	***9***	***19***	***36***
Less $^1/_3\,\mathbf{n}^3$	0	$^{-1}/_3$	$^{-8}/_3$	$^{-27}/_3$	$^{-64}/_3$	$^{-125}/_3$	$^{-216}/_3$
Gives	–1	$^{2}/_3$	$^{-2}/_3$	$^{-15}/_3$	$^{-37}/_3$	$^{-68}/_3$	$^{-108}/_3$

Because we've now introduced a $^1/_3$ fraction, just work through the rest in thirds. Also be very careful on signs and carry on with these new results, which be a quadratic.

0	**1**	**2**	**3**	**4**	**5**	**6**
–1	$^{2}/_3$	$^{-2}/_3$	$^{-15}/_3$	$^{-37}/_3$	$^{-68}/_3$	$^{-108}/_3$
	$^{5}/_3$	$^{-4}/_3$	$^{-13}/_3$	$^{-22}/_3$	$^{-31}/_3$	$^{-40}/_3$
		–3	–3	–3	–3	–3

The $?\mathbf{n}^2$ term is half of (–3) ie $({}^{-3}/_2\,\mathbf{n}^2)$.

So at $n = 1$ we have $^{-3}/_2 + ? - 1 = {}^2/_3$ so $? = {}^{19}/_6$

Put the whole lot together and we have $^1/_3\,\mathbf{n}^3 - {}^3/_2\,\mathbf{n}^2 + {}^{19}/_6\,\mathbf{n} - 1$

Actually that's not as bad as it looks because if we take out a factor $^1/_6$ we get

$$^1/_6\,(2\mathbf{n}^3 - 9\mathbf{n}^2 + 19\mathbf{n} - 6) \qquad \text{(which doesn't look too bad.)}$$

Factors of $^1/_3$ and $^1/_6$ are very common in cubics (why?) *rg*