Expectation

Definition

For a discrete function E[X] is defined as $\sum_{i=1}^{i=n} x_i p_i$

For a continuous function

$E[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx$

It is the generalised weighted mean function.

Hence E[X} = μ

The following relationships hold

$E[X + Y] = E[X] + E[Y]$

$E[aX] = a\, E[X]$

if $E[X] = 0$ then $X = 0$

if$X = Y$ then $e[X]= E[Y]$

if $X = k$ then $E[X] = k$

If X and Y are independent then

$E[XY] = E[X]\times E[Y]$

Variance

$$\begin{aligned} Var[X] &= E[\,(X - E[X])^2\,] \\ &= E[\,X^2 - 2X\,E[X] + E[X]^2 \\ &= E[X^2 - 2\,E[X]^2 + E[X]^2 \\ &= E[X^2] - E[X]^2 \end{aligned}$$

Minimum Value

We can choose any reference point instead of E [X] say k.

$$\begin{aligned} Var[X] &= E[\,(X - k)^2\,] \\ &= E[\,X^2 - 2\,k\,X + k^2\,] \\ &= E[X^2] - 2\,k\,E[X] + E[k]^2 \\ &= E[k]^2 - 2\,k\,E[X] + E[X^2] \end{aligned}$$

Now let

$A \Rightarrow E[k]^2 - 2\,k\,E[X]$ and

$B \Rightarrow E[X^2]$

Introduce the term $E[X]^2$ to each.

$A \Rightarrow E[k]^2 - 2\,k\,E[X] + E[X]^2$

$B \Rightarrow E[X^2] - E[X]^2$

Now we can factorise A to

$A \Rightarrow E[\,(k - E[X])^2\,]$

A is 0 when $k = E[X]$

ie $k = \mu$

and we are left with $E[X^2] - E[X]^2$ as before.

Hence the value of Var[X] is a minimum when $k = E[X]$.

Consequences

When we take a sample we use $\overline{x}$ as our best estimate of μ.

However this is unlikely to be exact and so our initial estimate of the variance is biased – it is at the idealised minimum.

Strictly we are "missing" one degree of freedom for the calculation of $\overline{x}$.

The rule is

"whenever we use $\overline{x}$ as an estimate of μ we must divide by $n-1$ instead of n".

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Consequences

When we take a sample we use $\bar{x}$ as our best estimate of μ.

However this is unlikely to be exact and so our initial estimate of the variance is biased – it is at the idealised minimum.

Strictly we are "missing" one degree of freedom for the calculation of $\bar{x}$.

The rule is

"whenever we use $\bar{x}$ as an estimate of μ we must divide by $n - 1$ instead of n".