

Volume of a Frustrum

A frustrum is a cone with the top chopped off. In practice it is a more commonly occurring shape than the cone eg a pile of sand or wheat is more like a frustrum than a cone. The formula for the volume is not given in the GCSE syllabus but it is implied that students should be able to calculate the volume from first principles. It's a tricky calculation. Here's two ways to find the general formula.

Method A

Let the height of a cone be t
 Chop the top off and let the remaining height be h
 Let the smaller radius be r
 Let the bigger radius be R

To calculate the volume, you find the volume of the whole cone less the volume of the chopped off bit.

$$\text{Volume} = \frac{1}{3} \pi R^2 t - \frac{1}{3} \pi r^2 (t-h)$$

Now from similar triangles it can be shown that $t = \frac{Rh}{R-r}$

Substituting back in we obtain

Volume

$$\begin{aligned} &= \frac{1}{3} \pi R^2 \left(\frac{Rh}{R-r} \right) - \frac{1}{3} \pi r^2 \left(\frac{Rh}{R-r} - h \right) \\ &= \frac{1}{3} \pi R^3 \frac{h}{R-r} - \frac{1}{3} \pi Rhr^2/R-r + \frac{1}{3} \pi r^2 h \\ &= \frac{1}{3} \pi h \left(\frac{R^3}{R-r} - \frac{Rr^2}{R-r} + r^2 \right) \\ &= \frac{1}{3} \pi h \left[\frac{R}{R-r} (R^2 - r^2) + r^2 \right] \\ &= \frac{1}{3} \pi h \left[\frac{R}{R-r} (R + r)(R - r) + r^2 \right] \\ &= \frac{1}{3} \pi h [R(R + r) + r^2] \\ &= \frac{1}{3} \pi h (R^2 + Rr + r^2) \end{aligned}$$

which is actually quite a neat looking formula.

Method B

Is it possible to derive the formula by a bit of logical deduction?

What is the simplest it could be?

$$\text{Start with Volume} = \frac{1}{3} \pi h R^2$$

Now widening out the top to a new radius r will increase the volume. We need to introduce r on equal terms to R because reversing R and r will invert the frustrum but not change the volume. So we deduce

$$\text{Volume} = \frac{1}{3} \pi h (R^2 + r^2) \quad ?\text{maybe}$$

Setting $r = 0$ gives us the volume of a cone again, so that's OK.

Setting $r = R$ should give us the volume of a cylinder.

$$\text{Volume} = \frac{1}{3} \pi h 2R^2 - \text{incorrect.}$$

That term $\frac{1}{3}$ at the beginning needs to be eliminated. So we need to add another squared term, involving R and r on equal terms. It has to be Rr .

So now we have

$$\text{Volume} = \frac{1}{3} \pi h (R^2 + Rr + r^2)$$

Which correctly reduces to

$$\text{Volume} = \frac{1}{3} \pi R^2 h \text{ when}$$

$r = 0$ (a cone) &

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So this formula is the simplest it could possibly be and give the correct results in the two limiting cases. ∞ rg

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