

Calculating Bernoulli Numbers

The recurrence formula is

$$\sum_{k=0}^{n-1} \binom{n}{k} B_k = 0 \text{ with seed } B_0 = 1$$

The practicality of this is that we set up the pseudo binomial expansion of $(B+1)^n$, eliminate the first term and equate what remains to zero. Substituting previously known B numbers determines the next.

$$\text{So } (B+1)^2 = B^2 + 2B + 1$$

$$\text{Let } 2B + 1 = 0 \text{ giving } B_1 = -1/2$$

$$(B+1)^3 = B^3 + 3B^2 + 3B + 1$$

$$\text{Set } 3B^2 + 3B + 1 = 0$$

$$\text{So } 3B^2 + 3(-1/2) + 1 = 0$$

$$\text{giving } B_2 = 1/6$$

$$(B+1)^4 = B^4 + 4B^3 + 6B^2 + 4B + 1$$

$$\text{Set } 4B^3 + 6B^2 + 4B + 1 = 0$$

$$\text{So } 4B^3 + 6(1/6) + 4(-1/2) + 1 = 0$$

$$\text{giving } B_3 = 0 \text{ perhaps surprisingly.}$$

$$(B+1)^5$$

$$= B^5 + 5B^4 + 10B^3 + 10B^2 + 5B + 1$$

$$\text{Set } 5B^4 + 10B^3 + 10B^2 + 5B + 1 = 0$$

$$\begin{aligned} \text{So } 5B_4 + 10(0) + 10(1/6) \\ + 5(-1/2) + 1 = 0 \end{aligned}$$

$$\text{giving } B_4 = -1/30$$

$$\begin{aligned} (B+1)^6 &= B^6 + 6B^5 + 15B^4 \\ &+ 20B^3 + 15B^2 + 6B + 1 \end{aligned}$$

$$\begin{aligned} \text{Set } 6B^5 + 15B^4 + 20B^3 \\ + 15B^2 + 6B + 1 = 0 \end{aligned}$$

$$\begin{aligned} \text{So } 6B_5 + 15(-1/30) + 20(0) \\ + 15(1/6) + 6(-1/2) + 1 = 0 \\ \text{giving } B_5 = 0 \end{aligned}$$

$$\begin{aligned} (B+1)^7 &= B^7 + 7B^6 + 21B^5 + 35B^4 + \\ &235B^3 + 21B^2 + 7B + 1 \end{aligned}$$

$$\begin{aligned} \text{Set } 7B^6 + 21B^5 + 35B^4 \\ + 35B^3 + 21B^2 + 7B + 1 = 0 \\ \text{So } 7B_6 + 21(0) + 35(-1/30) + 35(0) \\ + 21(1/6) + 7(-1/2) + 1 = 0 \\ \text{giving } B_6 = 1/42 \end{aligned}$$

$$\begin{aligned} (B+1)^8 &= B^8 + 8B^7 + 28B^6 + 56B^5 + \\ &70B^4 + 56B^3 + 28B^2 + 8B + 1 \end{aligned}$$

$$\begin{aligned} \text{Set } 8B^7 + 28B^6 + 56B^5 + 70B^4 + \\ 56B^3 + 28B^2 + 8B + 1 = 0 \\ \text{So } 8B_7 + 28(1/42) + 56(0) + 70(-1/30) \\ + 56(0) + 28(1/6) + 8(-1/2) + 1 = 0 \\ \text{giving } B_7 = 0 \text{ as expected} \end{aligned}$$

$$\begin{aligned} (B+1)^9 &= B^9 + 9B^8 + 36B^7 + 84B^6 \\ &+ 126B^5 + 126B^4 + 84B^3 \\ &+ 36B^2 + 9B + 1 = 0 \end{aligned}$$

From hereon I'm omitting the odd terms in the expansion

$$\text{So } 9B_8 + 84 \left(\frac{1}{42}\right) + 126 \left(-\frac{1}{30}\right) + 36 \left(\frac{1}{6}\right) + 9 \left(-\frac{1}{2}\right) + 1 = 0$$

$$\text{giving } B_8 = -\frac{1}{30}$$

B_9 is assumed to be = 0

$$\begin{aligned} (B+1)^{11} &= B^{11} + 11B^{10} + 55B^9 + 165B^8 + 330B^7 + 462B^6 + 462B^5 + 330B^4 \\ &\quad + 165B^3 + 55B^2 + 11B + 1 = 0 \end{aligned}$$

$$\begin{aligned} \text{So } 11B_{10} + 165 \left(-\frac{1}{30}\right) + 462 \left(\frac{1}{42}\right) + 330 \left(-\frac{1}{30}\right) + 55 \left(\frac{1}{6}\right) \\ + 11 \left(-\frac{1}{2}\right) + 1 = 0 \end{aligned}$$

$$\text{giving } B_{10} = \frac{5}{66}$$

$$\begin{aligned} (B+1)^{13} &= B^{13} + 13B^{12} + 78B^{11} + 286B^{10} + 715B^9 + 1287B^8 + 1716B^7 \\ &\quad + 1716B^6 + 1287B^5 + 715B^4 + 286B^3 + 78B^2 + 13B + 1 \\ &= 0 \end{aligned}$$

$$\begin{aligned} \text{So } 13B_{12} + 286 \left(\frac{5}{66}\right) + 1287 \left(-\frac{1}{30}\right) + 1716 \left(\frac{1}{42}\right) + 715 \left(-\frac{1}{30}\right) + 78 \left(\frac{1}{6}\right) \\ + 13 \left(-\frac{1}{2}\right) + 1 = 0 \end{aligned}$$

$$\text{giving } B_{12} = -\frac{691}{2730}$$

and truncating the initial expansion we have

$$\begin{aligned} 15B_{14} + 455 \left(-\frac{691}{2730}\right) + 3003 \left(\frac{5}{66}\right) + 6435 \left(-\frac{1}{30}\right) + 5005 \left(\frac{1}{42}\right) \\ + 1365 \left(-\frac{1}{30}\right) + 105 \left(\frac{1}{6}\right) + 15 \left(-\frac{1}{2}\right) + 1 = 0 \end{aligned}$$

$$\text{giving } B_{14} = \frac{7}{6}$$

which is perhaps also surprising

Patterns of Bernoulli Numbers

The following patterns are observed

B_n is always rational.

$$B_{2n+1} = 0 \text{ for } n \geq 1.$$

B_{2n} alternates sign

$$B_{4n} < 0 \text{ and } B_{4n+2} > 0 \text{ for } n \geq 1$$

The magnitude of B_{2n} grows very quickly

Occurrence of Bernoulli Numbers

In the formula for power series the Bernoulli number is always the last term.

$$\text{eg } \sum n^2 = \frac{1}{3} n^3 + \frac{1}{2} n^2 + \frac{1}{6} n$$

The formula for the squared power series ends in the 2nd Bernoulli number.

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