The Operator i

Draw a vector from (0,0) to (5,0)

Now define the operator i such that i is the instruction to rotate any object by 90° anticlockwise.

So our vector is now pointing from (0,0) to (0,5)

Now apply that operator again and the vector now point from

(0,0) to (–5,0)

We designate the total transfer at i^2

So we have rotated the point (5,0) to (–5,0)

So $5 \times i^2 = -5$

So $i^2 = -1$

Now strictly we have a quadratic equation and to solve it we take the square root of either side but remember to add the ± sign.

So $i = \pm \sqrt{-1}$

Most maths teachers will ignore the negative solution and simply tell you that $I = \sqrt{-1}$

This is incorrect. There are two errors.

The first error is that we began by describing an operator i that performed a specific transformation.

However we then manipulated that operator as an algebraic entity without any justification.

Secondly we then omitted one of the solutions when attempting to define i as a "number".

BY way of parallel we previously introduced the operator d/dy to determine the gradient of a function f(x) but having then derived the expression dy/dx immediately treat dy and dx as separate algebraic identities that can be manipulated as ordinary algebra.

Summary

Treat i as an operator but wherever i^2 occurs replace it with -1.

Otherwise leave it alone.

We define the complex plane by an Argand diagram where the x-axis becomes the Re (real) axis and the y-axis becomes the (Im (imaginary) axis. A complex number consists of a real part a and an imaginary part ib.

$$(a + ib) (c + id) = ac + aid + cib + i^2bd.$$

Using our substitution rule and collecting terms this becomes

$$(a + ib) (c + id) = (ac - bd) + i (bc + ad)$$

And that is the briefest introduction to imaginary and complex numbers.

The Paradox of i

Going right back to my university college notes i was simply introduced as $\sqrt{-1}$ and the implications of that described in detail.

It was only later when teaching higher maths at Taunton School that a colleague pointed out to me that the strict definition is $i^2 = -1$ and if we are going to engage in the fantasy of taking the square roots of both sides we need to add in the $\pm$ term because strictly we are now solving a quadratic equation.

An immediate consequence of the simple assumption that we can introduce the term $\sqrt{-1}$ is that it produces two solutions to all quadratic equations even when the discriminant is negative.

Suppose $b^2 - 4ac = -9$.

We write
$$\begin{aligned} \sqrt{-9} &= \sqrt{(-1 \times 9)} \\ &= \sqrt{-1} \times \sqrt{9} \\ &= i \times 3 \end{aligned}$$
and more commonly written 3i.

So far so good.

However if we then revisit the manipulation of square roots we already know that supposedly
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$

Example

$\sqrt{4} \times \sqrt{9} = 2 \times 3 = 6$
and $\sqrt{4 \times 9} = \sqrt{36} = 6$

But suppose we try
$$\begin{aligned} \sqrt{-4} \times \sqrt{-9} &= \sqrt{-4 \times -9} \\ &= \sqrt{36} \\ &= 6 \end{aligned}$$
Is this correct? *(Plot spoiler - it isn't)*

Going back to i is

$$\begin{aligned} \sqrt{-1} \times \sqrt{-1} &= \sqrt{(-1 \times -1)} \\ &= \sqrt{1} = 1 \end{aligned}$$

or is

$\sqrt{-1} \times \sqrt{-1} = i \times i = i^2 = -1$

The answer can't be both 1 and -1

The "answer" is that we are now told that $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ is only true for a and b both positive.

If they are not both positive we resort to the manipulation of i and substitute
$i^2 = -1$ at the end.

Yet in establishing that very process we actually wrote $\sqrt{-9} = \sqrt{(-1 \times 9)}$

So is the rule
$\sqrt{a} \times \sqrt{b} = \sqrt{ab}$
only when a and b are positive but

$\sqrt{-a} = \sqrt{(-1 \times a)} = \sqrt{(-1)} \times \sqrt{a}$
is still correct.

After 55 years I'm still confused.

Robert Goodhand

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