Multiplying Complex Numbers

Introduction

Let w = a + ib and z = c + id

Expressing in modulus / argument form

$w = r_1\theta$ and $z = r_2\phi$

where $r_1 = \sqrt{\{a^2 + b^2\}}$

and $r_2 = \sqrt{\{c^2 + d^2\}}$

Modulus of Products

So $w \times z = (a + ib)(c + id)$

$= (ac - bd) + i(bc + ad)$

So $|w \times z| = r_1 r_2$

$= \sqrt{\{(ac - bd)^2 + i(bc + ad)^2\}}$

Now $(ac - bd)^2 = a^2c^2 + b^2d^2 - 2abcd$

and $(bc + ad)^2 = b^2c^2 + a^2d^2 + 2abcd$

and the sum $= a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2$

So $r_1 r_2 = \sqrt{\{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2\}}$

But $r_1 = \sqrt{\{a^2 + b^2\}}$

and $r_2 = \sqrt{\{c^2 + d^2\}}$

So $r_1 r_2 = \sqrt{\{(a^2 + b^2)(c^2 + d^2)}$

$= \sqrt{\{a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2\}}$

Therefore $|w \times z| = |w| \times |z|$

So when multiplying two complex numbers the modulus of the product equals the product of the moduli.

Arguments of Products

$\theta = \tan^{-1} {}^b/_a$ and $\phi = \tan^{-1} {}^d/_c$

$\theta + \phi = \tan^{-1} {}^b/_a + \tan^{-1} {}^d/_c$

and assuming the given relationship for adding arctans

$= \tan^{-1}\{[\,{}^b/_a + {}^d/_c\,] / [\,1 - {}^b/_a \times {}^d/_c]$

and by multiplying through by ac

$= \tan^{-1}\{[\,bc + ad\,] / [\,ac - bd\,]\}$

which we can immediately see is the argument of the product.

So the argument of the product is the sum of the individual arguments.

Summary

if $w = r_1\theta$ and $z = r_2\phi$

then $w \times z = r_1 r_2 (\theta + \phi)$

rg

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So $r_1 r_2 = \sqrt{(a^2 + b^2)(c^2 + d^2)}$

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