Quadratics Worksheet

Introduction

For this exercise we need a graphical calculator – preferable a Texas TI–83 or TI–84.

The standard form of the quadratic is

$$ax^2 + bx + c.$$

In this worksheet we will make $a = 1$ to keep matters simple.

See the Appendix when $a > 1$.

So know we have the quadratic

$$x^2 + bx + c.$$

Exercise 1

Let's plot some basic graphs.

Try $\quad Y_1 = x^2$

What form the graph takes.

Now try $\quad Y_2 = 2x^2$ and

and $\quad Y_3 = 3x^2$

Erase these and experiment with

$$Y_1 = {}^-x^2$$
$$Y_2 = {}^-2x$$
$$Y_3 = {}^-3x^2$$

Erase these and now try

Try $\quad Y_1 = x^2 + 3$

and $\quad Y_2 = x^2 + 5$

and $\quad Y_3 = x^2 - 1$

So we can see that just as with

$$y = mx + c$$

changing the value of "c" moves the whole graph up and down.

Remember that the "m" value rotated the graph – changed the gradient. Can we do this with the quadratic?

No we can't. Quadratics only come in two forms – $\cup$ shaped and $\cap$ shaped. We can't rotate quadratics – strictly they wouldn't even be functions in a rotated form. But we can move them from side to side as well as up and down. But how?

Exercise 2

Let's give it some values

say $\quad x^2 - 3x - 5.$

We say that $f(x) = x^2 - 4x - 5$

The function of x is $x^2 - 4x - 5$

The term f(x) <u>for this question</u> encodes all the information about the function values "a", "b", and "c". It's like a piece of super algebra – "f" is now encoding several values rather than just one.

Erase the graphs and plot

$$Y_1 = x^2 - 4x - 5$$

In this pure algebraic form the one useful piece of information we can read straight off is the y-intercept – the point where the function crosses the y-axis. In this case it's $^-5$, the numerical coefficient "c".

But note the new position of the graph. Could we have predicted the position from the numerical coefficients "$^-4$" and "$^-5$"? Unlikely – or we wouldn't be doing this worksheet.

Our first task is to try and factorise

$$x^2 - 4x - 5.$$

Can we think of two factors of "5" that either have the sum or difference of "4".

Well we've made this super easy because as "5" is prime it only has two factors anyway "1" and "5" and their difference is "4".
So our factors are

$$f(x) = (x - 5) (x + 1)$$

Plot this expression for Y_2. We can only see one curve – one sits exactly on top of the other exactly.

Now go back and look very closely at our plotted graph Y_2.
Where are the values " $^{-}5$ " and "1"?
They are the coordinates where the curve crosses the x-axis.
They are the values of x in the expression $x^2 - 4x - 5$ that result in the function f(x) equating to zero.
These are called the **roots** of the equation – alternative known as the "solutions" or more simply "the zeros".

So the factorised version of the quadratic immediately gives us the roots. But why?
(ask your tutor to explain if necessary)

Exercise 3

There is one other "point" on the curve that particularly interests us – that is the "minimum" value or as AQA terms it "the turning point" – the bottom of the $\cup$. If we had plotted a "minus x^2" curve it would have a "maximum", the top of the $\cap$.
Where are the coordinates of that value hidden in the expression

$$x^2 - 4x - 5 \ ?$$

"Completing the Square"
This is tricky so follow carefully.
Take half the value of the "b" term – in this case "4" and subtract that from "x". So now we have $(x - 2)$.

Square that to get $(x - 2)^2$.

Now multiply that out.

$$(x - 2)^2 = (x - 2) (x - 2)$$
$$= x^2 - 4x + 4$$

So we've managed to create the first two terms of our original expression but the third term is "wrong" – we've got a "+4" and we want a "$^{-}5$".

Well no matter – let's plough on regardless and force a fit by subtraction the 4 to get rid of it and then subtracting 5. So now we have

$$x^2 - 4x + 4 - 4 - 5$$
$$= (x - 2)^2. - 9$$

giving the final expression
To be clear what's happening plot the three functions back into the calculator.

$$Y_1 = x^2 - 4x - 5$$
$$Y_2 = (x - 5) (x + 1)$$
$$Y_3 = (x - 2)^2 - 9$$

We only see one single curve because these three expressions are all identical. Y_2 and Y_3 just express the original function in different forms. Yet each form tells us something different about the quadratic. Y_1 gives the y-intercept, Y_2 the roots and Y_3 the coordinates of the minimum.

Where are those integers "2" and "9" in the curve? Well they are the coordinates of the minimum value – though annoyingly the "sign" of the x value changed form a "minus" to a "plus". The coordinates are (2, $^-9$)

Now plot $Y_4 = x^2$.
Confirm that we have shifted that original x^2 function 2 units to the right and 9 units down.

Tutor Exercise

Ask your tutor to show you how the expression $(x – 2)^2 – 9$ encodes the roots – 5 and +1.

Exercise 4

Clear the calculator.
Enter $Y_1 = x^2$.
Suppose we'd like to move that function 4 units to the right and 3 units down. Try this

$$Y_2 = (x – 4)^2 – 3.$$

Sure enough, we've shifted the curve as required. Now multiply out the expression and enter against Y_3.

$Y_3 = x^2 – 8x + 13$

Again Y_2 and Y_3 line up exactly and we've moved the curve from its original central position 4 units to the right and 3 units down.

The Rules

To move a function up and down change the constant – the "c" term to that value. Change "c" to "+3" and the quadratic obligingly moves "3" units "up".

To move the function say "3" units to the right change every value of x to (x – 3).

To move the function say "3" units to the left change every value of x to (x + 3).

Also note that the minimum (or maximum) value lies midway between the roots.

But why is there that annoying change of sign? Unfortunately this worksheet is too small to contain that – but your tutor may be able to help by looking back to y = mx + c.

Appendix

Plot $Y_1 = 2x^2 – 16x + 26$
Divide through by 2.
Plot $Y_2 = x^2 – 8x + 13$
What do we notice?
Hint – have the roots changed?

So if we want to solve $ax^2 + bx + c$ first we divide through by "a" to produce $x^2 + {}^b/_a\, x + {}^c/_a$.

Although we've changed the shape of the function we haven't changed the roots.

Appendix 2
Deriving the General Formula

Let $f(x) = ax^2 + bx + c$

Step 1

Divide through by a gives

$$g(x) = x^2 + {}^{b}/_{a}\, x + {}^{c}/_{a}.$$

where $g(x)$ has the same roots as $f(x)$.

Step 2

We halve the "x" term, subtract from "x", and "square" to give

$(x + {}^{b}/_{2a})^2 = x + {}^{b}/_{a} x + {}^{b^2}/_{4a^2}$

So we conclude

$x^2 + {}^{b}/_{a}\, x + {}^{c}/_{a}. = (x + {}^{b}/_{2a})^2 - {}^{b^2}/_{4a} + {}^{c}/_{a}$

Step 3

Now when $g(x) = 0$

$$\begin{aligned} (x + {}^{b}/_{2a})^2 &= {}^{b^2}/_{4a^2} - {}^{c}/_{a} \\ &= {}^{b^2}/_{4a^2} - {}^{4ac}/_{4a^2} \\ &= {}^{(b^2 - 4ac)} / {}_{4a^2} \end{aligned}$$

so $(x + {}^{b}/_{2a}) = {}^{\pm\sqrt{(b^2 - 4ac)}} / {}_{2a}$

so $\qquad x = {}^{\{- b \pm\sqrt{(b^2 - 4ac)}\}} / {}_{2a}$

To be clear these are the two values of x that are the roots or solutions of the function $g(x)$ and the original function $f(x)$ ie the x-value for

$$f(x) = 0.$$

Features of the Quadratic

The term $(b^2 - 4ac)$ is termed the discriminant Δ. There are 3 cases

1) When $\Delta > 0$ there are two roots
2) When $\Delta = 0$ there is just one root – the quadratic minimum sits right on the x-coordinate.
eg clear and enter $Y_1 = (x - 3)^2$
Then $(x - 3)^2 = x^2 - 6x + 9$ and sure enough $6^2 = 4 \times 1 \times 9$ $(b^2 = 4ac)$
Try $Y_1 = 4x^2 - 9$
and $4x^2 - 9 = (2x + 3) (2x - 3)$
This is called the difference of two perfect squares

3) When $\Delta < 0$ the quadratic sits above the x coordinate and there are no real roots to the function.

The vertex of the quadratic is the vertical line that passes through the minimum or maximum of the quadratic and sits at $x = {}^{-b}/{}_{2a}$ #
The roots of the quadratic sit symmetrically either side of the vertex at a distance.

$$\pm\sqrt{(b^2 - 4ac)} / 2a$$

When there is no "b" term, the vertex is just the y-axis as $x = 0$

Conclusion

So now the roots of the quadratic given by

$$\{^{-} b \pm\sqrt{(b^2 - 4ac)}\} / 2a$$

make complete sense – yes?

For those who know their calculus this can be derived immediately by differentiating $ax^2 + bx + c$ and equating to zero.