Mr. G's Handbook of Mathematics

Differentiation and Integration Standard Forms

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Definition of Differentiation

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Limit}} \frac{f(x+\delta x) - f(x)}{\delta x} \quad †$$

and it follows that $\frac{dy}{dx} = 1 / \frac{dx}{dy}$

Chain Rule *(function of a function f(x) and g(x))*

$$\frac{d}{dx}[f\,g(x)] = f'[g(x)] \bullet g'(x)$$

or more simply $\frac{dy}{dx} = \frac{dy}{du} \bullet \frac{du}{dx}$

Product Rule *(Leibnitz Rule)*

$$\frac{d}{dx}(uv) = v\frac{du}{dx} + u\frac{dv}{dx}$$

Generalised Power Rule *(where u and v are functions of x)*

$$\frac{d}{dx}(u^v) \qquad vu^{(v-1)} \bullet \frac{du}{dx} + u^v \bullet \ln u \bullet \frac{dv}{dx}$$

To derive take logs of both sides and then differentiate using the Product Rule

Power Rule *(where u = x and v = n)*

$$\frac{d}{dx}(x^n) = n\,x^{n-1}$$

The power rule can be derived directly from logarithmic differentiation $\frac{d}{dx} \ln x^n$

Quotient Rule *(derived from the Product Rule)*

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \left[v\frac{du}{dx} - u\frac{dv}{dx}\right] / v^2$$

Reciprocal Rule *(case of Quotient Rule u = 1)*

$$\frac{d}{dx}\left(\frac{1}{v}\right) = \frac{-dv/dx}{v^2}$$

Logarithmic Rule *(case of Chain rule where u = ln x)*

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$$

Stationary Points

if $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} > 0$ *then we have a minimum point*

if $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} < 0$ *then we have a maximum point*

if $\frac{dy}{dx} = 0$ *and* $\frac{d^2y}{dx^2} = 0$ *then the situation is indeterminate*

Notes

† *Non standard analysis of hyperreals legitimises the infinitesimal* dx *without using limits.*

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Integration - Primary Methods

$\int (u + v + w)\, dx = \int u dx + \int v dx + \int w dx$ *terms can be integrated separately*

$\int a\, f(x) dx = a \int f(x) dx$ *constants unaffected by integration*

$\int f(x)\, dx = \int f(u)\, {}^{dx}/_{du}\, du$ *Integration by substitution*

$\int u\, dv = uv - \int v\, du$ *Integration by parts* †

Let u *be the easy differential and* v *be the easy integral.*

If there is just an unintegrable function u *then multiply by 1 (strictly you set* $dv = 1 \cdot dx$ *).*

$\int f'(x)[f(x)]^n dx = ({}^1/_{n+1})\, [f(x)]^{n+1}$ *when integral is of this pattern*

Trigonometric Identities

To integrate $\sin^2 x\, dx$ *or* $\cos^2 x\, dx$ *write in terms of* $\cos 2x$

To integrate $\tan^2 x\, dx$ *write in terms of* $\sec^2 x$

To integrate $\cot^2 x\, dx$ *write in terms of* $\operatorname{cosec}^2 x$

Definite Integrals

$\int_a^b f'(x)\, dx = f(b) - f(a)$

$\text{Area} = \int_a^b f(x)\, dx$

$\text{Area} = \int_a^b f(y_1 - y_2)\, dx$

see also trapezium rule in numerical methods

volume rotation $= \pi \int_a^b y^2\, dx$

Integrals involving Natural Logs.

Remember that $\ln a = {}^{-}\ln ({}^1/_a)$.

In this booklet only the positive log solution is given.

Where log solutions are involved, the absolute value should be specified.

Notes

† *This follows immediately from* $d(uv) = v du + u dv$

For repeated application, differentiate down first column and integrate down second.

Answer is the product of the diagonals alternately + then − then + etc.

Standard Differentiations and Integrations

$\int f(x)\,dx = ? + c$	$f(x)$	$f'(x) = {}^{dy}/_{dx}$
$-\cos\theta$	$\sin\theta$	$\cos\theta$
$\sin\theta$	$\cos\theta$	$-\sin\theta$
$\ln\lvert\sec\theta\rvert$	$\tan\theta$	$\sec^2\theta \equiv 1 + \tan^2\theta$
$\ln\lvert\operatorname{cosec}\theta - \cot\theta\rvert$	$\operatorname{cosec}\theta$	$-\operatorname{cosec}\theta \bullet \cot\theta$
$\ln\lvert\sec\theta + \tan\theta\rvert$ †	$\sec\theta$	$\sec\theta \bullet \tan\theta$
$\ln\lvert\sin\theta\rvert$	$\cot\theta$	$-\operatorname{cosec}^2\theta$
$x\sin^{-1}x + \sqrt{(1-x^2)}$	$\sin^{-1}x$	${}^{1}/_{\sqrt{(1-x^2)}}$
$x\cos^{-1}x - \sqrt{(1-x^2)}$	$\cos^{-1}x$	$-{}^{1}/_{\sqrt{(1-x^2)}}$
$x\tan^{-1}x - \frac{1}{2}\ln\lvert 1 + x^2\rvert$	$\tan^{-1}x$	${}^{1}/_{(1+x^2)}$
$x\operatorname{cosec}^{-1}x + \cosh^{-1}x$	$\operatorname{cosec}^{-1}x$	$-{}^{1}/_{(\lvert x\rvert\sqrt{(x^2-1)})}$
$x\sec^{-1}x - \cosh^{-1}x$	$\sec^{-1}x$	${}^{1}/_{(\lvert x\rvert\sqrt{(x^2-1)})}$
$x\cot^{-1}x + \frac{1}{2}\ln\lvert 1 + x^2\rvert$	$\cot^{-1}x$	$-{}^{1}/_{(1+x^2)}$
$\cosh x$	$\sinh x$	$\cosh x$
$\sinh x$	$\cosh x$	$\sinh x$
$\ln\lvert\cosh x\rvert$	$\tanh x$	$\operatorname{sech}^2 x \equiv 1 - \tanh^2 x$
$\ln\lvert\coth x - \operatorname{cosech} x\rvert$	$\operatorname{cosech} x$	$-\operatorname{cosech} x \bullet \coth x$
$\tan^{-1}(\sinh x)$ †	$\operatorname{sech} x$	$-\operatorname{sech} x \bullet \tanh x$
$\ln\lvert\sinh x\rvert$	$\coth x$	$-\operatorname{cosech}^2 x$
$x\sinh^{-1}x - \sqrt{(x^2+1)}$	$\sinh^{-1}x$	${}^{1}/_{\sqrt{(x^2+1)}}$
$x\cosh^{-1}x - \sqrt{(x^2-1)}$	$\cosh^{-1}x$	${}^{1}/_{\sqrt{(x^2-1)}}$
$x\tanh^{-1}x + \frac{1}{2}\ln\lvert 1 - x^2)\rvert$	$\tanh^{-1}x$	${}^{1}/_{(1-x^2)}$ $\quad \lvert x\rvert < 1$
$x\operatorname{cosech}^{-1}x + \sinh^{-1}x$	$\operatorname{cosech}^{-1}x$	$-{}^{1}/_{(\lvert x\rvert\sqrt{(1+x^2)})}$
$x\operatorname{sech}^{-1}x + \sin^{-1}x$	$\operatorname{sech}^{-1}x$	$-{}^{1}/_{x\sqrt{(1-x^2)}}$
$x\coth^{-1}x + \frac{1}{2}\ln\lvert x^2 - 1\rvert$	$\coth^{-1}x$	${}^{1}/_{(1-x^2)}$ $\quad \lvert x\rvert > 1$

Notes

† *see notes on the Gudermannian function in the Hyperbolic Booklet*

the constant of integration is assumed.

Standard Differentiations and Integrations $x \to ax + b$

$\int f(x)\,dx = ? + c$	$f(x)$	$f'(x) = {}^{dy}/_{dx}$	
$-({}^1/_a)\cos(a\theta + b)$	$\sin(a\theta + b)$	$a\cos(a\theta + b)$	
$({}^1/_a)\sin(a\theta + b)$	$\cos(a\theta + b)$	$-a\sin(a\theta + b)$	
$({}^1/_a)\ln\lvert\sec(a\theta)\rvert$	$\tan(a\theta + b)$	$a\sec^2(a\theta + b)$	
$({}^1/_a)\ln\lvert\mathrm{cosec}(a\theta) - \cot(a\theta)\rvert$	$\mathrm{cosec}(a\theta + b)$	$-a\,\mathrm{cosec}(a\theta+b) \bullet \cot(a\theta+b)$	
$({}^1/_a)\ln\lvert\sec(a\theta) + \tan(a\theta)\rvert$	$\sec(a\theta + b)$	$a\sec(a\theta + b) \bullet \tan(a\theta+b)$	
$({}^1/_a)\ln\lvert\sin(a\theta)\rvert$	$\cot(a\theta + b)$	$-a\,\mathrm{cosec}^2(a\theta + b)$	
$x\sin^{-1}{}^x/_a + \sqrt{(a^2 - x^2)}$	$\sin^{-1}{}^x/_a$	${}^1/_{\sqrt{(a^2 - x^2)}}$	$x^2 < a^2 \quad a > 0$
$x\cos^{-1}{}^x/_a - \sqrt{(a^2 - x^2)}$	$\cos^{-1}{}^x/_a$	$-{}^1/_{\sqrt{(a^2 - x^2)}}$	
$x\tan^{-1}{}^x/_a - \tfrac{1}{2}a\ln\lvert a^2+x^2\rvert$	$\tan^{-1}{}^x/_a$	${}^a/_{(x^2 + a^2)}$	
$x\,\mathrm{cosec}^{-1}{}^x/_a + a\cosh^{-1}{}^x/_a$	$\mathrm{cosec}^{-1}{}^x/_a$	$-{}^a/_{\lvert x\rvert\sqrt{(x^2 - a^2)}}$	
$x\sec^{-1}{}^x/_a - a\cosh^{-1}{}^x/_a$	$\sec^{-1}{}^x/_a$	${}^a/_{\lvert x\rvert\sqrt{(x^2 - a^2)}}$	
$x\cot^{-1}{}^x/_a + \tfrac{1}{2}a\ln\lvert a^2+x^2\rvert$	$\cot^{-1}{}^x/_a$	$-{}^a/_{(x^2 + a^2)}$	
$({}^1/_a)\cosh ax$	$\sinh(ax)$	$a\cosh(ax)$	
$({}^1/_a)\sinh ax$	$\cosh(ax)$	$a\sinh(ax)$	
$({}^1/_a)\ln\lvert\cosh ax\rvert$	$\tanh(ax)$	$a\,\mathrm{sech}^2(ax)$	
$({}^1/_a)\ln\lvert\tanh(\tfrac{1}{2}ax)\rvert$	$\mathrm{cosech}(ax)$	$-a\,\mathrm{cosech}(ax) \bullet \coth(ax)$	
$({}^1/_a)\,2\tan^{-1}(e^{ax})$	$\mathrm{sech}(ax)$	$-a\,\mathrm{sech}(ax) \bullet \tanh(ax)$	
$({}^1/_a)\ln(\sinh ax)$	$\coth(ax)$	$-a\,\mathrm{cosech}^2(ax)$	
$x\sinh^{-1}{}^x/_a - \sqrt{(x^2 + a^2)}$	$\sinh^{-1}{}^x/_a$	${}^1/_{\sqrt{(x^2 + a^2)}}$	$a > 0$
$x\cosh^{-1}{}^x/_a - \sqrt{(x^2 - a^2)}$	$\cosh^{-1}{}^x/_a$	${}^1/_{\sqrt{(x^2 - a^2)}}$	$x > a \quad a > 0$
$x\tanh^{-1}{}^x/_a + \tfrac{1}{2}a\ln\lvert a^2 - x^2\rvert$	$\tanh^{-1}{}^x/_a$	${}^a/_{(a^2 - x^2)}$ †	$x^2 < a^2 \quad a > 0$
$x\,\mathrm{cosech}^{-1}{}^x/_a + a\sinh^{-1}{}^x/_a$	$\mathrm{cosech}^{-1}{}^x/_a$	${}^{-a}/_{x\sqrt{(x^2 + a^2)}}$	
$x\,\mathrm{sech}^{-1}{}^x/_a - a\cos^{-1}{}^x/_a$	$\mathrm{sech}^{-1}{}^x/_a$	${}^{-a}/_{x\sqrt{(a^2 - x^2)}}$	
$x\coth^{-1}{}^x/_a + \tfrac{1}{2}\ln\lvert x^2 - a^2\rvert$	$\coth^{-1}{}^x/_a$	${}^a/_{(a^2 - x^2)}$ †	$x^2 > a^2$

Notes

† *The differentials appear identical, but remember the functions have no common mapping.*

Log expressions of integrands can be tidied up by $\times$ $\div$ *through by any constant.*

Standard Differentiations and Integrations

$\int f(x)\,dx = ? + c$	$f(x)$	$f'(x) = {}^{dy}/_{dx}$
$\frac{1}{2} ax^2 + bx$	$ax + b$	a
${}^1/_{(n+1)} \cdot x^{n+1}$	x^n	$n x^{n-1}$
e^x	e^x	e^x
$-e^{-x}$	e^{-x}	$-e^{-x}$
no general integral.	$e^{f(x)}$	$f'(x)\, e^{f(x)}$
$x \ln\lvert x\rvert - x$	$\ln \lvert x\rvert$ †	${}^1/_x$
$({}^1/_{\ln w})(x \ln\lvert x\rvert - x)$	$\log_w \lvert x\rvert$ ‡	${}^1/_{x.\ln w}$
no general integral except when $f(x) = x^n$	$\ln f(x)$	${}^{f'(x)}/_{f(x)}$
$\frac{1}{2} x^2 \ln\lvert x\rvert - \frac{1}{4} x^2$	$x \ln\lvert x\rvert$	$1 + \ln x$
$\ln(\ln\lvert x\rvert)$	${}^1/_{x \ln \lvert x\rvert}$	$-{}^1/_{x^2 \ln x} - {}^1/_{x^2 (\ln x)^2}$
$({}^1/_{\ln\lvert w\rvert})w^x$	w^x	$w^x (\ln w)$ $w>0$
$-({}^1/_{\ln\lvert w\rvert})w^{-x}$	w^{-x}	$-\ln w \times w^{-x}$
not integrable	$w^{(1/x)}$	$-({}^1/_{x^2}) \ln w \times w^{(1/x)}$
not integrable	$w^{-(1/x)}$	$({}^1/_{x^2}) \ln w \times w^{-(1/x)}$
not integrable	x^x	$(\ln x + 1) \times x^x$
not integrable	x^{-x}	$-(\ln x + 1) \times x^{-x}$
not integrable	$x^{1/x}$	$({}^1/_{x^2})(1 - \ln x)\, x^{1/x}$
not integrable	$x^{-1/x}$	$({}^1/_{x^2})(\ln x - 1)\, x^{-1/x}$

Other Useful Integrals

$\sqrt{(x + a)}\ ({}^2/_3)(x + 2a)$	${}^x/_{\sqrt{(x+a)}}$	*use substitution* $u = \sqrt{(x + a)}$
$\sqrt{(x - a)}\ ({}^2/_3)(x - 2a)$	${}^x/_{\sqrt{(x-a)}}$	*use substitution* $u = \sqrt{(x - a)}$
$\sqrt{(a - x)}\ ({}^2/_3)(2a - x)$	${}^x/_{\sqrt{(a-x)}}$	*use substitution* $u = \sqrt{(a - x)}$

Notes

† *Some authorities define* $\ln x$ *to be* $\int_1^x {}^1/_t\, dt$

‡ *think of this as* ${}^{\ln x}/_{\ln w}$

Standard Differentiations and Integrations $x \to ax + b$

$\int f(x)\,dx = ? + c$	$f(x)$	$f'(x) = {}^{dy}/_{dx}$
$\frac{1}{2} ax^2 + bx$	$ax + b$	a
$({}^1/_a)({}^1/_{n+1})(ax+b)^{n+1}$	$(ax+b)^n$	$an(ax+b)^{n-1}$
$({}^1/_a) . e^{ax+b}$	e^{ax+b}	ae^{ax+b}
$-({}^1/_a) . e^{-(ax+b)}$	$e^{-(ax+b)}$	$-a e^{-(ax+b)}$
no general form	$e^{f(x)}$	$f'(x) e^{f(x)}$
$(x + {}^b/_a) . \ln\|ax+b\| - x$	$\ln\|ax+b\|$	${}^a/_{(ax+b)}$ *and if* $b=0$ *then* ${}^1/_x$
$({}^1/_{\ln w})\{(x+{}^b/_a).\ln(ax+b) - x\}$	$\log_w\|ax+b\|$	$({}^1/_{\ln w})\, {}^a/_{(ax+b)}$
$-{}^1/_{f(x)}$	$f'(x)/_{[f(x)]^2}$	
$\frac{1}{2}(x^2 - {}^{b^2}/_{a^2})\ln\|ax+b\| - {}^1/_{4a}x(ax-2b)$	$x\ln\|ax+b\|$	$\ln\|ax+b\| + {}^{ax}/_{ax+b}$
$2\sqrt{f(x)}$	$f'(x)/_{\sqrt{f(x)}}$	*used to integrate some inverse functions*
$({}^1/_{a\ln\|w\|})\, w^{ax+b}$	w^{ax+b}	$\ln w \bullet a\, w^{ax+b}$
$-{}^1/_{\ln\|w\|}\, w^{-x}$	$w^{-(ax+b)}$	$-\ln w \bullet a\, w^{-(ax+b)}$
no integral?	$w^{(1/ax+b)}$	$-({}^1/_{(ax+b)^2}) \ln w \bullet a\, w^{(1/ax+b)}$
no integral?	$w^{-(1/x)}$	$({}^1/_{(ax+b)^2}) \ln w \bullet a\, w^{(-1/ax+b)}$
no integral?	x^{ax+b}	$(ax\ln x + ax + b) \bullet x^{ax+b-1}$
no integral?	$x^{-(ax+b)}$	$-(ax\ln x + ax + b) \bullet x^{-(ax+b)-1}$
no integral?	$x^{1/ax+b}$	$({}^1/_{(ax+b)^2}).(a\,x\ln x - ax - b)\, x^{1/ax+b-1}$
no integral?	$x^{-1/ax+b}$	$({}^1/_{(ax+b)^2}).(a\,x\ln x - ax - b)\, x^{-1/ax+b-1}$

Some Tricky Integrals (using inverse trig substitutions)

$\frac{1}{2}[a^2 \sinh^{-1} {}^x/_a + x\sqrt{(x^2+a^2)}]$	$\sqrt{(x^2+a^2)}$	
$\frac{1}{2}[-a^2 \cosh^{-1} {}^x/_a + x\sqrt{(x^2-a^2)}]$	$\sqrt{(x^2-a^2)}$	
$\frac{1}{2}[-a^2 \cos^{-1} {}^x/_a + x\sqrt{(a^2-x^2)}]$	$\sqrt{(a^2-x^2)}$	*depending which substitution used*
$\frac{1}{2}[a^2 \sin^{-1} {}^x/_a + x\sqrt{(a^2-x^2)}]$	$\sqrt{(a^2-x^2)}$	*will differ by* $\frac{1}{4}a^2\pi$
$\frac{1}{2}[-a^2 \sinh^{-1} {}^x/_a + x\sqrt{(x^2+a^2)}]$	${}^{x^2}/_{\sqrt{(x^2+a^2)}}$	
$\frac{1}{2}[a^2 \cosh^{-1} {}^x/_a + x\sqrt{(x^2-a^2)}]$	${}^{x^2}/_{\sqrt{(x^2-a^2)}}$	
$\frac{1}{2}[-a^2 \cos^{-1} {}^x/_a - x\sqrt{(a^2-x^2)}]$	${}^{x^2}/_{\sqrt{(a^2-x^2)}}$	*depending which substitution used*
$\frac{1}{2}[a^2 \sin^{-1} {}^x/_a - x\sqrt{(a^2-x^2)}]$	${}^{x^2}/_{\sqrt{(a^2-x^2)}}$	*will differ by* $\frac{1}{4}a^2\pi$

Additional Differentiations and Integrations

$\int f(\theta)\, d\theta = ? + c$	$f(\theta)$	$f'(\theta) = {}^{dy}/_{d\theta}$
$\theta/2 - 1/4 \sin(2\theta)$	$\sin^2\theta$	$2 \sin\theta \cos\theta$ †
$\theta/2 + 1/4 \sin(2\theta)$	$\cos^2\theta$	$-2 \sin\theta \cos\theta$
$\tan\theta - \theta$	$\tan^2\theta$	$2 \tan\theta \sec^2\theta$
$-\cot\theta$	$\mathrm{cosec}^2\theta$	$-2\, \mathrm{cosec}^2\theta \cot\theta$
$\tan\theta$	$\sec^2\theta$	$2 \sec^2\theta \tan\theta$
$-\cot(2\theta) - \theta$	$\cot^2\theta$	$-2 \cot\theta\, \mathrm{cosec}^2\theta$
$\sin\theta - \theta\cos\theta$	$\theta\sin\theta$	$\theta\cos\theta + \sin\theta$
$\cos\theta + \theta\sin\theta$	$\theta\cos\theta$	$\cos\theta - \theta\sin\theta$
No integral	$\theta\tan\theta$	$\theta\sec^2\theta + \tan\theta$
No integral	$\theta\, \mathrm{cosec}\,\theta$	$\mathrm{cosec}\,\theta\,(1 - \theta\cot\theta)$
No integral	$\theta\sec\theta$	$\sec\theta\,(1 + \theta\tan\theta)$
No integral	$\theta\cot\theta$	$-\theta\, \mathrm{cosec}^2\theta + \cot\theta$
$-x/2 + 1/4 \sinh(2x)$	$\sinh^2 x$	$2 \sinh x \cosh x$
$x/2 + 1/4 \sinh(2x)$	$\cosh^2 x$	$2 \sinh x \cosh x$
$x - \tanh x\theta$	$\tanh^2 x$	$2 \tanh x\,(1 - \tanh^2 x)$
$\coth x$	$\mathrm{cosech}^2 x$	$-2\, \mathrm{cosech}^2 x \coth x$
$\tanh x$	$\mathrm{sech}^2 x$	$-2\, \mathrm{sech}^2 x \tanh x$
$x - \coth x$	$\coth^2 x$	$-2 \coth x\, \mathrm{cosech}^2 x$
$-\sinh x + x\cosh x$	$x\sinh x$	$x\cosh x + \sinh x$
$-\cosh x + x\sinh x$	$x\cosh x$	$\cosh x + x\sinh x$
No integral	$x\tanh x$	$x\, \mathrm{sech}^2 x + \tanh x$
No integral	$x\, \mathrm{cosech}\, x$	$\mathrm{cosech}\, x\,(1 - x\coth x)$
No integral	$x\, \mathrm{sech}\, x$	$\mathrm{sech}\, x\,(1 - x\tanh x)$
No integral	$x\coth x$	$-x\, \mathrm{cosech}^2 x + \coth x$

Notes

† = $\sin 2\theta$

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Reduction Formulae *(derived from Integration by Parts)*

Let I_n	=	$\int \sin^n x \, dx$	=	$-\frac{1}{n} \sin^{n-1} x \cdot \cos x + \frac{n-1}{n} I_{n-2}$
Let I_n	=	$\int \cos^n x \, dx$	=	$\frac{1}{n} \cos^{n-1} x \cdot \sin x + \frac{(n-1)}{n} I_{n-2}$
Let I_n	=	$\int \tan^n x \, dx$	=	$\frac{1}{n-1} \tan^{n-1} x - I_{n-2}$
Let I_n	=	$\int \mathrm{cosec}^n x \, dx$	=	$-\frac{1}{n-1} \mathrm{cosec}^{n-2} x \cdot \cot x + \frac{n-2}{n-1} I_{n-2}$
Let I_n	=	$\int \sec^n x \, dx$	=	$\frac{1}{n-1} \sec^{n-2} x \cdot \tan x + \frac{n-2}{n-1} I_{n-2}$
Let I_n	=	$\int \cot^n x \, dx$	=	$-\frac{1}{n-1} \cot^{n-1} x - I_{n-2}$
Let I_n	=	$\int \sinh^n x \, dx$	=	$\frac{1}{n} \sinh^{n-1} x \cdot \cosh x - \frac{(n-1)}{n} I_{n-2}$
Let I_n	=	$\int \cosh^n x \, dx$	=	$\frac{1}{n} \cosh^{n-1} x \cdot \sinh x - \frac{(n-1)}{n} I_{n-2}$
Let I_n	=	$\int \tanh^n x \, dx$	=	$-\frac{1}{n-1} \tanh^{n-1} x + I_{n-2}$
Let I_n	=	$\int \mathrm{cosech}^n x \, dx$	=	$-\frac{1}{n-1} \mathrm{cosech}^{n-2} x \cdot \coth x - \frac{n-2}{n-1} I_{n-2}$
Let I_n	=	$\int \mathrm{sech}^n x \, dx$	=	$\frac{1}{n-1} \mathrm{sech}^{n-2} x \cdot \tanh x + \frac{n-2}{n-1} I_{n-2}$
Let I_n	=	$\int \coth^n x \, dx$	=	$-\frac{1}{n-1} \coth^{n-1} x + I_{n-2}$
Let I_n	=	$\int x^n e^x \, dx$	=	$x^n e^x - n I_{n-1}$
Let I_n	=	$\int (\ln x)^n \, dx$	=	$x (\ln x)^n - n I_{n-1}$

Example layout when using Reduction Formulae

$\int x^3 e^x \, dx$	=	$x^3 e^x - 3 I_2$	
I_2	=	$x^2 e^x - 2 I_1$	
I_1	=	$x^1 e^x - 2 I_0$	
I_0	=	e^x	*then rebuild step by step*
I_1	=	$x^1 e^x - e^x$	
I_2	=	$x^2 e^x - 2xe^x + 2e^x$	
I_3	=	$x^3 e^x - 3x^2e^x + 6xe^x - 6e^x$	
	=	$e^x(x^3 - 3x^2 + 6x - 6)$	

Notes

also note $\int x^n e^{ax} \, dx = \frac{1}{a} \left(x^n e^{ax} - n \int x^{n-1} e^{ax} \, dx\right)$

Solution First Order Linear Differential Equations

Standard form is $\frac{dy}{dx} + P(x)y = Q(x)$

The solution is given by first determining the integrating factor given by $e^{\int P dx}$

The transformed equation is then $y \cdot e^{\int P dx} = Q(x) \cdot e^{\int P dx}$

Bernouilli Differential Equations

These are of the form

$$\frac{dy}{dx} + P(x)y = Q(x) \cdot y^n$$

This is a non linear equation but is transformed to a linear equation by multiplying through by

$$y^{-n}(1-n)$$

This transforms y and produces a modified first order linear differential equation

A Few Standard Integrating Factors

$$e^{\int 1 dx} = e^x$$

$$e^{\int 2 dx} = e^{2x}$$

$$e^{\int 3 dx} = e^{3x}$$

$e\wedge(\int {}^1/_x) = x$ $\qquad$ $e\wedge(\int {}^{-1}/_x) = {}^1/_x$

$e\wedge(\int {}^2/_x) = x^2$ $\qquad$ $e\wedge(\int {}^{-2}/_x) = {}^1/_{x^2}$

$e\wedge(\int {}^3/_x) = x^3$ $\qquad$ $e\wedge(\int {}^{-3}/_x) = {}^1/_{x^3}$

$e\wedge(\int {}^1/_x) = x$ $\qquad$ $e\wedge(\int {}^{-1}/_x) = x^{-1}$

$e\wedge(\int {}^1/_{2x}) = x^{1/2}$ $\qquad$ $e\wedge(\int {}^{-1}/_{2x}) = x^{-1/2}$

$e\wedge(\int {}^1/_{4x}) = x^{1/4}$ $\qquad$ $e\wedge(\int {}^{-1}/_{4x}) = x^{-1/4}$

$e^{\int \tan x\, dx} = \sec x$ $\qquad$ $e^{\int \tanh x\, dx} = \cosh x$

$e^{\int \mathrm{cosec}\, x\, dx} = \mathrm{cosec}\, x - \cot x$ $\qquad$ $e^{\int \mathrm{cosech}\, x\, dx} = {}^1/_{\mathrm{cosech}\, x + \coth x}$

$e^{\int \sec x\, dx} = \sec x + \tan x$ $\qquad$ *doesn't have a neat hyperbolic equivalent*

$e^{\int \cot x\, dx} = \sin x$ $\qquad$ $e^{\int \coth x\, dx} = \sinh x$

Solution Second Order Linear Differential Equations

if $a\,{}^{d^2y}/_{dx^2} + b\,{}^{dy}/_{dx} + cy = 0$

The auxilary quadratic equation is $am^2 + bm + c = 0$.

When the auxilary equation has two real roots α *and* β *the general solution is*

$$y = Ae^{\alpha x} + Be^{\beta x}$$

When the auxilary equation has one coincident root α

$$y = (A + Bx)e^{\alpha x}$$

When the auxilary equation has the conjugate roots $p \pm iq$

$$y = e^{px}(A \cos qx + B \sin qx)$$

Forms of Particular Integral

if $a\,{}^{d^2y}/_{dx^2} + b\,{}^{dy}/_{dx} + cy = f(x)$ $\qquad y =$ *compl. func. + particular integral*

for $f(x)$	=	k	*set*	y	=	K
for $f(x)$	=	k *and* $c = 0$	*set*	y	=	Kx
for $f(x)$	=	px	*set*	y	=	$Px + Q$
for $f(x)$	=	$px + q$	*set*	y	=	$Px + Q$
for $f(x)$	=	$px^2 + qx + r$	*set*	y	=	$Px^2 + Qx + R$
for $f(x)$	=	ke^{qx}	*set*	y	=	Ke^{qx} *(or try* Kxe^{qx}*)*
for $f(x)$	=	$\cos kx$	*set*	y	=	$P \cos kx + Q \sin kx$
for $f(x)$	=	$\sin kx$	*set*	y	=	$P \cos kx + Q \sin kx$
for $f(x)$	=	$\cos kx + \sin kx$	*set*	y	=	$P \cos kx + Q \sin kx$
if $f(x)$	=	$\cos k_1x + \sin k_2x$	*who knows?*			

Differentiation of Pcos kx + Qsin kx

$$y = P \cos kx + Q \sin kx$$

$${}^{dy}/_{dx} = -k P \sin kx + k Q \cos kx$$

$${}^{d^2y}/_{dx^2} = -k^2 P \cos kx - k^2 Q \sin kx$$

Full Solutions of First Order Differential Equations

$$\frac{dy}{dt} = 2t \quad \textit{then} \quad y = t^2 + y_0$$

$$\frac{dy}{dt} = cy \quad \textit{then} \quad y = y_0\, e^{ct}$$

$$a\frac{dy}{dt} + by = 0 \quad \textit{then} \quad y = y_0\, e^{-(b/a)t}$$

$$a\frac{dy}{dt} + by + c = 0 \quad \textit{then} \quad y = \frac{-c}{a} + \left(y_0 + \frac{c}{a}\right)e^{-(b/a)t}$$

$$a\frac{dy}{dt} + by + c = c\,\sin\omega t \quad \textit{then} \quad y = \frac{c\,\sin(\omega t - \phi)}{\sqrt{(a^2 + \omega^2 b^2)}}$$

$$\sin\phi = \frac{\omega b}{\sqrt{(a^2 + w^2 b^2)}}$$

Full Solutions of Second Order Differential Equations

$$\frac{d^2y}{dt^2} = a \quad \textit{then} \quad y = \tfrac{1}{2}at^2 + y_1 t + y_0$$

$$\textit{cf} \quad s = ut + \tfrac{1}{2}at^2$$

$$\frac{d^2y}{dt^2} = n^2 y \quad \textit{then} \quad y = \tfrac{1}{2}\left[\left(y_0 + \frac{y_1}{n}\right)e^{mt} + \left(y_0 - \frac{y_1}{n}\right)e^{-nt}\right]$$

$$= y_o \cosh nt + \frac{y_1}{m}\sinh nt$$

$$\frac{d^2y}{dt^2} = {}^{-}n^2 y \quad \textit{then} \quad y = y_o \cos nt + \frac{y_1}{m}\sin nt$$

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \quad \textit{then} \quad y = e^{-mt}\left(y_o \cosh nt + \frac{y_1 + m}{n}\sinh nt\right)$$

$$m = \frac{b}{2a}$$

$$n = \frac{\sqrt{(b^2 - 4ac)}}{2a}$$

an alternative representation is $y = e^{-mt}(Ae^{nt} + Be^{-nt})$

where $A = \tfrac{1}{2}[\,y_0 - (\,y_1 + my_0\,)\,]$ †

where $B = \tfrac{1}{2}[\,y_0\,(\,y_1 + my_0\,) - y_0\,]$ †

I can also write this equation as a second order system response (this takes me back!)

$$\frac{d^2y}{dt^2} + 2\xi\omega_n\frac{dy}{dt} + w_n^2 y = 0 \quad \textit{then} \quad y = e^{-\xi\omega t} R\,\sin\left(w_n t\sqrt{(1-\xi^2)} + \phi\right)$$

$$\textit{where} \quad R = \sqrt{\left\{\left[\frac{(y1+y0)}{wn\sqrt{(1-\xi^2)}}\right]^2 + y_0^2\right\}}$$

$$\textit{and} \quad \phi = \tan^{-1}\left\{\left(\frac{1}{R}\right)\frac{(yo\omega n\sqrt{(1-\xi^2)})}{y1} + y_0\right\}$$

Notes

† *I determined these by taking Laplace transforms and rearranging to an alternative form.*

Special Integrals

The Error Function

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$$

The probability density function of the Normal Distribution - the constant to make area = 1

The Elliptic Integral

$$\int_0^\theta \sqrt{(1 - k^2 \sin^2\theta)}\,d\theta$$

Encountered when attempting to calculate length of arc of an ellipse.

The Gamma Function

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$$

For $n \in Z^+$ *then* $\Gamma(n+1) = n!$ *ie the gamma function is the generalised factorial function.*

The Beta Function

$$B(n,m) = \int_0^1 x^{n-1}(1-x)^{m-1}dx$$

$$B(n,m) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

The beta function can itself be used to evaluate integrals such as $\int_0^1 (1 - x^4)^{-1/2}\,dx$

The Bessel Function

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\,d\theta$$

The Bessel function is a function of x *and* n *and satifies the Bessel equation order* n

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{n^2}{x^2}\right)y = 0$$

Bessel functions are important in the advanced study of high-frequency electrical conductors.

The Logarithmic Integral

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\ln t}\,dt$$

Used in estimates of asymptotic values, including estimate of prime counting function $\pi(x)$.

The Cosine Integral

$$\mathrm{Ci}(x) = \int_x^\infty \frac{\cos t}{t}\,dt$$ *plus its hyperbolic partner*

Used in many applications including signal processing, semiconductor and quantum physics.

The Sine Integral

$$\mathrm{Si}(x) \quad \int_x^\infty \frac{\sin t}{t}\,dt$$ *plus its hyperbolic partner*

Used in many applications including signal processing, semiconductor and quantum physics.

The Exponential Integral

$$\mathrm{Ei}(x) \quad \int_x^\infty e^{-t} / t\,dt$$

The integral of any function of the form $R(x)e^x$, *where* R *is a rational function, reduces to an elementary integral and* $\mathrm{Ei}(x)$

Mr. G´s Integration Tips

Is the function in a standard form?

eg one of the trig functions,

exponential function, log function, algebraic etc.

Is the integrand of the form $f'(x) / f(x)$?

$f'(x) / f(x)$ then integral will be of the form $\ln | f(x) |$

Is the integrand of the form $f'(x) [f(x)]^n$?

Just integrate $[f(x)]^n$, differentiate your answer and adjust.

Can the integrand be written in the form $^{ax + b} / _{(x + c) (x + d)}$?

Use partial fractions to simplify into separate fractions and use standard forms to resolve.

Is the integrand a product of two distinct functions?

Use integration by parts $\int u \, dv = uv - \int v \, du$

Choose u for the function that can be easily differentiated and reduced to a constant.

Is the integrand a trig function that can be replaced by a standard expression?

Know your standard trig identities.

Can you simplify the integral by changing the variable?

Simplify the integrand with a substitution.

rg

Counting

No.	***Greek***	***Latin***
1	*mono*	*uni*
2	*duo*	*bi*
3	*tri*	*tri*
4	*tetra*	*quad*
5	*penta*	*quin*
6	*hexa*	*sex*
7	*hepta*	*sept*
8	*octo*	*oct*
9	*nona*	*non*
10	*deca*	*dec*

The "Handbook of Mathematics" has been written and produced by Robert Goodhand

Although the formulae and expressions given have been individually derived and checked errors do creep in. Sections of the Handbook are continuously updated.

If you would like the latest issue, just email me at robert.goodhand@gmail.com

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